✍1 Show that $(I - \beta v v^T)x = \alpha e_1$ when $v = x - \alpha e_1$ and $\alpha = \pm\|x\|_2$.

Solution: Equivalent to showing that

$$x - (\beta x^T v)v = \alpha e_1 \quad \text{i.e.,} \quad x - \alpha e_1 = (\beta x^T v)v$$

but recall that $v = x - \alpha e_1$ so we need to show that

$$\beta x^T v = 1 \quad \text{i.e., that} \quad \frac{2}{\|x - \alpha e_1\|_2^2}(x^T v) = 1$$

- Denominator = $\|x\|_2^2 + \alpha^2 - 2\alpha e_1^T x = 2(\|x\|_2^2 - \alpha e_1^T x)$
- Numerator $= 2x^T v = 2x^T(x - \alpha e_1) = 2(\|x\|_2^2 - \alpha x^T e_1)$

Numerator/ Denominator = 1. □

✍2 $Pw = ?$

Solution: $Pw = -w$

□

✍3 Cost of Householder QR?

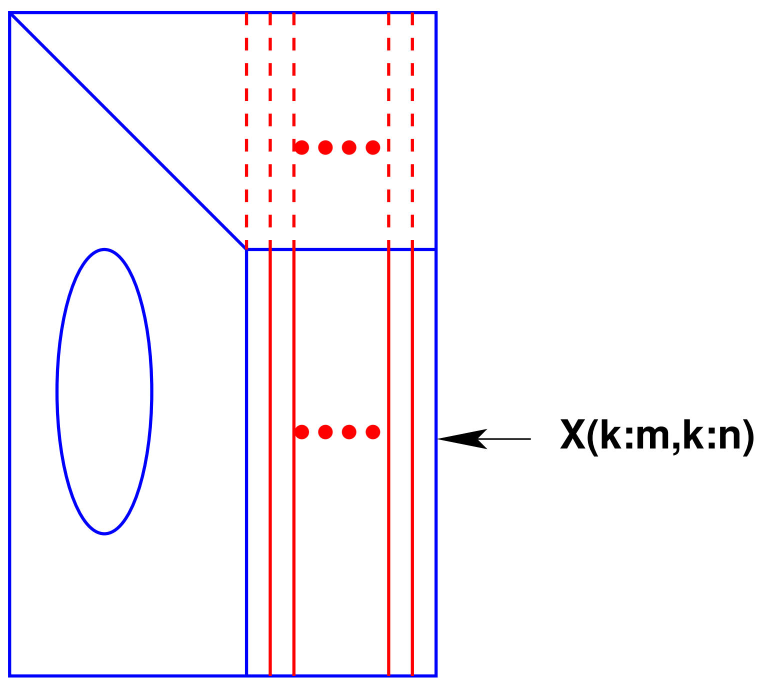
Solution: Look at the algorithm: each step works in rectangle $X(k : m, k : n)$. Step k : twice $2(m-k+1)(n-k+1)$

$$
\begin{aligned}
T(n) &= \sum_{k=1}^{n} 4(m-k+1)(n-k+1) \\
&= 4\sum_{k=1}^{n}[(m-n)+(n-k+1)](n-k+1) \\
&= 4[(m-n)*\frac{n(n+1)}{2}+\frac{n(n+1)(2n+1)}{6}] \\
&\approx (m-n)*2n^2+4n^3/3 \\
&= 2mn^2-\frac{2}{3}n^3
\end{aligned}
$$

Recall that for Gram-Scmidt we had a cost of $\approx 2mn^2$. The difference can be significant (Householder faster) when n is not that small relative with m. $\square$

✍4 Suppose you know the norms of each column of X at the start. What happens to each of the norms of $X(2 : m, j)$ for $j = 2, \cdots, n$? Generalize this to step k and obtain a procedure to inexpensively compute the desired norms at each step.

Solution: The trick that is used is that *the 2-norm of each column does not change thoughout the algorithm.* This is simple to see because each column is multiplied by a Householder transformation P_k at each step. These Householder transformations are unitary and preserve the length. The square of the 2-norm of $X(k:n,j)$ (solid red lines in Figure) is the original square of the 2-norm of $X(k:n,j)$ minus the square of the 2-norm of $X(1:k-1,j)$ (dashed red lines in Figure). (solid red lines in Figure) In order to *update* $\|X(k:n,j)\|^2$ – all we have to do is subtract $|X(k-1,j)|^2$ at each step k. This costs very little. $\square$



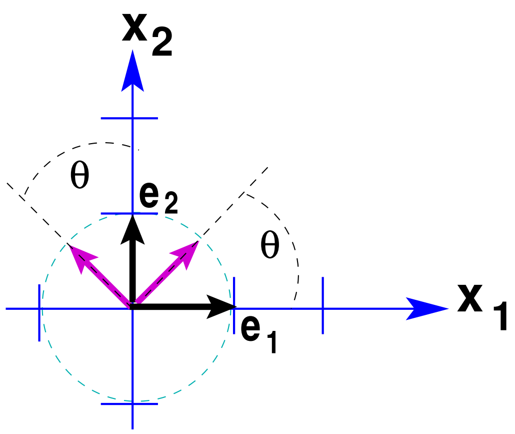
✍5 Consider the mapping that sends any point x in $\mathbb{R}^2$ into a point y in $\mathbb{R}^2$ that is rotated from x by an angle θ. Find the matrix representing the mapping. [Hint: observe how the canonical basis is transformed.] Show an illustration. What is the mapping correspoding to an angle $-\theta$?

Solution: The vector $e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is transformed to $\begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$. The vector $e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ is transformed to $\begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$.

These are the first and second columns of the mapping! So the matrix representing the rotation is

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

An illustration is shown in the figure.



A Givens rotation performs a rotation of angle $-\theta$. $\square$