✍1 How would you get an orthonormal basis of X?

Solution: Just take any basis and orthonormalize it with Gram-Schmidt of Householder QR. ☐

✍2 Show how you can get a decomposition in which C is lower (or upper) triangular, from the above factorization.

Solution: You first get any factorization in the form shown in Page 9-7 – Then to get an upper triangular C you use the QR factorization $C = QR$. Then U is replaced by

$$U_{new} = U \times \begin{pmatrix} Q & 0 \\ 0 & I \end{pmatrix}$$

and C is replaced by R. To get a lower triangular C you can use the same trick applied to A^T and transpose the final result. ☐

✍3 How can you get the ULV decomposition by using only the Householder QR factorization (possibly with pivoting)?

Solution: You first get the Householder QR factorization $A = Q_1 R_1$ of the matrix A. The second step is to perform a Householder QR

factorization of the matrix R_1^T, so you will get: $R_1^T = Q_2 R_2$. The final step is to write:

$$A = Q_1 * R_2^T * Q_2^T \equiv U R V^T$$

where $U = Q_1 \in \mathbb{R}^{m \times m}; V = Q_2 \in \mathbb{R}^{n \times n}; R = R_2 \in \mathbb{R}^{m \times n}$

$\square$

✐4 In the proof of the SVD decomposition, define U, V as single Householder reflectors.

Solution: We deal with U only [proceed similarly with V]. We need a matrix $P = I - 2ww^T$ such that the first column of A is u_1 and all columns are orthonormal. The second requirement is satisfied by default since P is unitary. Note that if P is available we will have $Pu_1 = e_1$ because $P^2 = I$. Therefore, the wanted w is simply the vector that transforms the vector u_1 into αe_1. ... $\square$

✐5 How can you obtain the thin SVD from the QR factorization of A and the SVD of an $n \times n$ matrix?

Solution: We first get the thin QR factorization of A, namely $A = QR$ where $Q \in \mathbb{R}^{m \times n}$ and $R \in \mathbb{R}^{n \times n}$; Then we can get the SVD

$R = U_R \Sigma V_R^T$ of R and this yields:

$$A = Q \times U_R \Sigma_R V_R^T \rightarrow A = U \Sigma V^T, \quad \text{with} \quad U = Q \times U_R; \quad \Sigma = \Sigma_R;$$