✍1 What is the inverse of a unitary (complex) or orthogonal (real) matrix?

Solution: If Q is unitary then $Q^{-1} = Q^H$. $\square$

✍2 What can you say about the diagonal entries of a skew-symmetric (real) matrix?

Solution: They must be equal to zero. $\square$

✍3 What can you say about the diagonal entries of a Hermitian (complex) matrix?

Solution: We must have $a_{ii} = \bar{a}_{ii}$. Therefore a_{ii} must be real. $\square$

✍4 What can you say about the diagonal entries of a skew-Hermitian (complex) matrix?

Solution: We must have $a_{ii} = -\bar{a}_{ii}$. Therefore a_{ii} must be purely imaginary. $\square$

✍5 Which matrices of the following type are also normal: real symmetric, real skew-symmetric, Hermitian, skew-Hermitian, complex symmetric, complex skew-symmetric matrices.

Solution: Real symmetric, real skew-symmetric, Hermitian, skew-Hermitian matrices are normal. Complex symmetric, complex skew-symmetric matrices are not necessarily normal. $\square$

✍6 Find all real 2×2 matrices that are normal.

Solution: [result only] You will find that all such matrices are of the form $I + \alpha B$ where α is a real scalar and B is either Symmetric or skew-symmetric. $\square$

✍7 Show that a triangular matrix that is normal is diagonal.

Solution: To simplify notation, we consider only the case of real matrices. We will use an induction argument on n this size of the matrix. The case $n = 1$ is trivial. Assume that the result is true for matrices of size $n - 1$ and let R be an upper triang. matrix of size n that is normal.

Since R us normal we have $R^T R = RR^T$. Let C be this product and consider the term c_{11}. Because $R^T R = RR^T$ we have on the one hand:

$$c_{11} = r_{11}^2$$

and on the other:

$$c_{11} = r_{11}^2 + r_{12}^2 + r_{13}^2 + \cdots + r_{1n}^2$$

By equating the two quantities we obtain:

$$r_{12}^2 + r_{13}^2 + \cdots + r_{1n}^2 = 0,$$

which implies that $r_{1j} = 0$ for $j > 1$, i.e., the entries of the first row of R - not including the diagonal - are all zero. The remaining matrix, namely $R_1 = R(2:n, 2:n)$ in matlab notation is a matrix of size $n-1$ and it can be seen that it satisfies the relation $R_1^T R_1 = R_1 R_1^T$ - because of the fact that $r_{1j} = 0$ for $j > 1$. Now our induction hypothesis will help us complete the proof since it implies that R_1 is diagonal. $\square$

✍9 What does the matrix-vector product Va represent?

Solution: If $a = [a_0, a_2, \cdots, a_n]$ and $p(t)$ is the n-th degree polymomial:

$$p(t) = a_0 + a_1 t + a_2 t^2 + \cdots a_n t^n$$

then Va is a vector whose components are the values $p(x_0)$, $p(x_1)$, $\cdots, p(x_n)$. $\square$

✍10 Interpret the solution of the linear system $Va = y$ where a is the unknown. Sketch a 'fast' solution method based on this.

Solution: Given the previous exercise, the interpretation is that we are seeking a polynomial of degree n whose values at $x_0, \cdots, x_n$ are the components of the vector y, i.e., $y_0, y_1, \cdots, y_n$. This is known as polynomial interpolation (see csci 5302). The polynomial can be determined by, e.g., the Newton table in $O(n^2)$ operations. $\square$

✍14 If C is circulant (real) and symmetric, what can be said about the c_is?

Solution: By comparing the first row and 1st column of C, one can see that when C is symmetric then the 1st row starting in position 2, i.e., the row $c(2 : n) = [c_2, ..., c_n]$ must be 'symmetric' in that $c_2 = c_n; c_3 = c_{n-1}; \cdots c_j = c_{n-j+2}; ..$ $\square$

✍15 What is the result of multiplying S_n by a vector? What are the powers of S_n? What is the inverse of S_n?

Solution: We consider the case $n = 5$ (no loss of generality). The vector $S_5 v$ results from v by shifting v cyclically upward. For the same reason, S_5^k shifts the columns of S_k upward cyclically k times. The inverse of S_5 corresponds to the inverse operation from 'shifting up', which is shifting down. The corresponding matrix, which is the inverse of S_5, is the transpose of S_5.

✍16 Show that

$$C = c_1 I + c_2 S_n + c_3 S_3^2 + \cdots + c_n S_n^{n-1}$$

As a result show that all circulant matrices of the same size commute.

Solution: The first term is indeed the diagonal of C. The second term is the diagonal matrix $c_2 I$ with entries shifted up (cyclically) by one position. The 3rd term is the diagonal matrix $c_3 I$ with entries shifted up (cyclically) by two positions, etc. This is indeed what is observed in C.

The product of two circulant matrices is a product like $p(S_n)q(S_n)$ where p, q are 2 polynomials of degree $n-1$. It is easy to see that for any matrix A, the products $p(A)q(A)$ and $q(A)p(A)$ are the same, which shows the result. $\square$

✍17 (Continuation) Use the result of the previous exercise to show that the product of two circulant matrices is circulant.

Solution: This is because $p(S_n)q(S_n)$ can be expressed as a polynomial of degree $n-1$ of S_n. Indeed note that $S_n^n = I$ so all powers in the product can be reduced to a power $< n$. $\square$

$\boxed{✍ 18}$ How many mode n fibers are there?

Solution: A mode n fiber is obtained by fixing all indices except the index in dimension n - resulting in a vector of dimension d_n (n-th dimension): $a(i_1, i_2, \cdots, i_{n-1}, :, i_{n+1}, \cdots, i_N)$

There are $d_1 \times d_2 \times d_{n-1} \times d_{n+1} \times \cdots d_N$ of selecting a fiber. For example for a 3-way tensor there are $d_2 d_3$ mode-1 fibers, $d_1 d_3$ mode-2 fibers, and $d_1 d_2$ mode-3 fibers. $\square$