UNIVERSITY OF MINNESOTA TWIN CITIES

CSCI 5304 Fall 2025

COMPUTATIONAL ASPECTS OF MATRIX THEORY

Class time	:	*TTh 8:15 – 9:30 am*
Room	:	*Lind Hall 302*
Instructor	:	*Yousef Saad*

Lecture notes: on Canvas and *http://www-users.cse.umn.edu/~saad/csci5304/*

September 2, 2025

About this class

- Instructor and Teaching Assistant:

➤ Me: Yousef Saad

➤ TA: Zechen Zhang

- Course titled: "Computational Aspects of Matrix Theory." Aims to cover:

"Everything that you can learn in one semester about computations that involve matrices, from theory, to algorithms, matlab/Python implementations, and applications."

➤ Subject is at the core of *most* disciplines requiring numerical computing..

➤ .. and gaining importance in Computer Science (machine learning, robotics, graphics, ...)

Objectives of this course

Set 1 Fundamentals of matrix theory :

- Matrices, subspaces, eigenvalues & eigenvectors
- Norms, matrix norms, understanding errors, sensitivity [somewhat theoretical]
- Various decompositions, LU, QR, SVD, ..

set 2 Computational linear algebra / Algorithms

- Solving linear systems, LU factorization
- Solving least-squares problems, QR factorization
- Eigenvalue problems - computing eigenvalues, eigenvectors,

Set 3 Linear algebra in applications: examples from computer science

Logistics:

- Canvas will be used to post everything relevant to the class, from Lecture notes, syllabus, schedule, a matlab/Python folder, etc.

- I also keep basic information (primarily: Lecture notes and lecture demos) in:

http://www-users.cse.umn.edu/~saad/csci5304

➤ The two sites have links that point to each other but you need only to rely on:

Canvas

Please Note:

- Homeworks, tests, and their solutions are copyrighted

  - *Solutions of HWs and tests are provided to you [Canvas] with the hope of helping you understand the material better. By accessing them you agree not to send them to others, sell them (%#!!$), or otherwise (help) make them available via external web-sites.*

About lecture notes:

- Lecture notes (like this first set) will be posted on the class web-site – usually before the lecture. Note: format of notes used in class may be slightly different from the one posted – but contents are identical.
- Review them to get some understanding if possible before class.
- Read the relevant section (s) in one of the references listed
- Lecture note sets are grouped by topics rather than by lecture.
- In the notes the symbol ✍1 indicates suggested easy exercises or questions sometimes solved in class.
- Each set will include a supplement with solutions to some of these exercises + possibly additional notes/comments (an evolving document)

About tests

➤ There are 6 quizzes and 3 midterms scheduled.

- Quizzes (10mn – 15mn at end of class) are generally on course material $\rightarrow$ No documents of any sort allowed.
- Mid-terms are 75mn (whole lecture session) - Formula sheet (2 pages) allowed.
- For both Quizzes and Mid-terms: lowest score is dropped
- No make-up quizzes or mid-terms.

Matlab and/or Python

- We will use matlab and/or Python+numpy to test algorithms (Demos)
- For assignments: Your codes can be in either Matlab or in Python+numpy (your choice)
- Some documentation for matlab is posted in the (class) matlab folder
- Important: I post the matlab diaries used for the demos (if any)...
- ... something similar for Python [under IPython]

- If you do not know matlab at all and have difficulties with it - and you do not know python - talk to me or the TA at office hours. You may need some initial help to get you started with matlab. Important point: this is *not* a programming course.

Final remarks on lecture notes

➤ Please do not hesitate to report errors and/or provide feedback on content.

➤ On occasion I will repost lecture notes with changes/additions

How to study for this course:

1) Rely primarily on lecture notes as a starting point. Use other sources [e.g., (online) books] to get a deeper understanding. Note: Quizzes will be strongly related to lecture notes (so you absolutely need to understand these at the minimum)

2) Do and redo the practice exercises done in class and those of the lecture notes. Participate in solving the exercises [at tests no one will be there to help you!]

3) Ask questions! Participate in discussions (office hours, canvas, ...)

Illness, Zoom, Office Hours, etc

- Classes are all in-person. I will use Zoom only when necessary (sickness, travel).
- If you are sick *please* do not come to class
- If I get sick - I will schedule the class on Zoom [Assuming I can!] –
- Office hours: See posted information for details (schedule, zoom option, etc.)
- Questions before we begin?

INTRODUCTION & BACKGROUND

- General Background: Linear algebra and numerical linear algebra
- Mathematical background: matrices, eigenvalues, rank, ...
- Types of matrices, structutred matrices, special matrices
- Review of Determinants (brief)

Introduction

➤ This course is about *Matrix algorithms* or "matrix computations"

➤ It involves: algorithms for standard matrix computations (e.g. solving linear systems) - and their analysis (e.g., their cost, numerical behavior, ..)

➤ Matrix algorithms pervade most areas of science and engineering.

➤ In computer science: recent increase of interest in matrix algorithms for data mining, information retrieval, search engines, pattern recognition, graphics, ...

General Problems in Numerical Linear Algebra (dense & sparse)

- Linear systems: $Ax = b$. Often: A is large and sparse
- Least-squares problems $\min \|b - Ax\|_2$
- Eigenvalue problem $Ax = \lambda x$. Several variations -
- SVD .. and
- ... Low-rank approximation, subspace approximation, etc.
- Tensors and low-rank tensor approximation
- Matrix equations: Sylvester, Lyapunov, Riccati, ..
- Nonlinear equations – acceleration methods
- Matrix functions and applications
- Many many more ...

Background in linear algebra

➤ Review vector spaces.

➤ A vector subspace of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that is also a real vector space. The set of all linear combinations of a set of vectors $G = \{a_1, a_2, \ldots, a_q\}$ of $\mathbb{R}^n$ is a vector subspace called the linear span of G,

➤ If the a_i's are linearly independent, then each vector of $\mathbf{span}\{G\}$ admits a unique expression as a linear combination of the a_i's. The set G is then called a *basis*.

✍1 Recommended reading: Sections 1.1 – 1.6 of

www.cs.umn.edu/~saad/eig_book_2ndEd.pdf

Matrices

➤ A real $m \times n$ matrix A is an $m \times n$ array of real numbers

$$a_{ij}, \quad i = 1, \ldots, m, \; j = 1, \ldots, n.$$

Set of $m \times n$ matrices is a real vector space denoted by $\mathbb{R}^{m \times n}$.

➤ Complex matrices defined similarly.

➤ A matrix represents a linear mapping between two vector spaces of finite dimension n and m:

$$x \in \mathbb{R}^n \quad \longrightarrow \quad y = Ax \in \mathbb{R}^m$$

➤ Recall: this mapping is linear [what does it mean?]

➤ Recall: Any linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$ *is* a matrix vector product

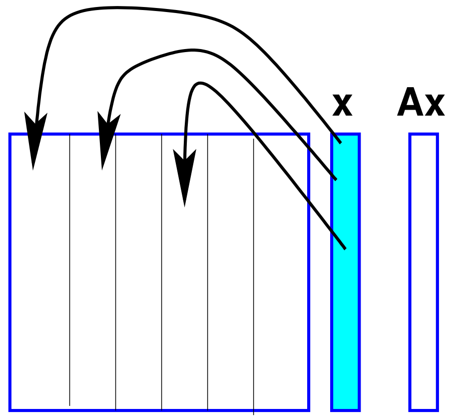
Matrix-vector product: computing $y = Ax$ (a review)

Matrix-vector products represent linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$

- $A \in \mathbb{R}^{m \times n}, \quad x \in \mathbb{R}^n, \quad y = Ax \in \mathbb{R}^m$

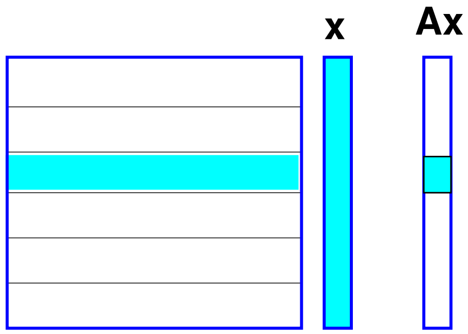
Column-form: $y = \sum_{j=1}^{m} x_j A(:, j)$

y = Linear combination of columns $A(:, j)$ with coefficients x_j



Row-form: $y_i = \sum_{k=1}^{n} a_{ik} x_k$

y_i = Dot product of $A(i, :)$ and x



Operations with matrices:

Addition: $C = A + B$, where $A, B, C \in \mathbb{R}^{m \times n}$ and

$$c_{ij} = a_{ij} + b_{ij}, \quad i = 1, 2, \ldots m, \quad j = 1, 2, \ldots n.$$

Multiplication by a scalar: $C = \alpha A$, where

$$c_{ij} = \alpha \, a_{ij}, \quad i = 1, 2, \ldots m, \quad j = 1, 2, \ldots n.$$

Multiplication by another matrix: $C = AB$,

where $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p}, C \in \mathbb{R}^{m \times p}$, and

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$

Transposition: If $A \in \mathbb{R}^{m\times n}$ then its transpose is a matrix $C \in \mathbb{R}^{n\times m}$ with entries

$$c_{ij} = a_{ji}, i = 1, \ldots, n, \; j = 1, \ldots, m$$

Notation : A^T.

Transpose Conjugate: for complex matrices, the transpose conjugate (or Hermitian transpose) matrix denoted by A^H is more relevant: $A^H = \bar{A}^T = \overline{A^T}$.

✍2 $(A^T)^T = ??$

✍3 $(AB)^T = ??$

✍4 $(A^H)^H = ??$

✍5 $(A^H)^T = ??$

✍6 $(ABC)^T = ??$

✍7 True/False: $(AB)C = A(BC)$

✍8 True/False: $AB = BA$

✍9 True/False: $AA^T = A^TA$

➤ Matlab notation - often used in this course:

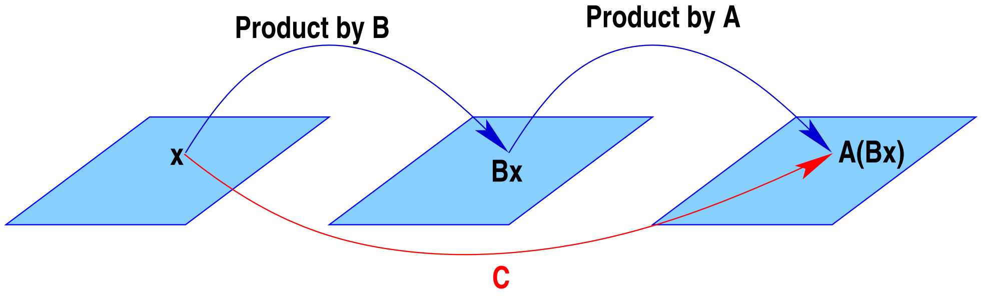
$A_{:,j}$ or $A(:, j)$ == j-the column of A

$A_{i,:}$ or $A(i, :)$ == i-th row of A

Review: Matrix-matrix products

- Recall definition of $C = A \times B$:
$(A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{m\times p})$

$$c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}.$$



- Recall: C represents a composition of mappings

- Can do the product column-wise

$$C_{:,j} = \sum_{k=1}^{n} b_{kj} A_{:,k}$$

- Can do it row-wise:

$$C_{i,:} = \sum_{k=1}^{n} a_{ik} B_{k,:}$$

➤ Can do it as a sum of 'outer-product' matrices:

$$C = \sum_{k=1}^{n} A_{:,k} B_{k,:}$$

✍10 Verify all 3 formulas above..

✍11 Complexity? [number of multiplications and additions]

✍12 What happens to these 3 different approches to matrix-matrix multiplication when B has one column $(p = 1)$?

✍13 Characterize the matrices AA^T and A^TA when A is of dimension $n \times 1$.

Kronecker products of matrices

➤ This is a special product of matrices that can be quite useful in some situations

Definition

For $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{p\times q}$ define:

(A matrix of size $(mp) \times (nq)$).

➤ In Matlab: `kron (A,B)`

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \cdots & \cdots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

➤ Note that the dimensions m, n, p, q, can be any (> 0) integers.

✍14 Show that for 2 vectors u, v we have $v^T \otimes u = uv^T$ and also that $u \otimes v^T = uv^T$

➤ The Kronecker sum of matrices also arises in some applications. If $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{m\times m}$ then their Kronecker sum is: $A \oplus B = A \otimes I_{m,m} + I_{n,n} \otimes B$

Range and null space (for $A \in \mathbb{R}^{m\times n}$)

- Range: $\mathrm{Ran}(A) = \{Ax \mid x \in \mathbb{R}^n\}$ $\subseteq \mathbb{R}^m$
- Null Space: $\mathrm{Null(A)} = \{x \in \mathbb{R}^n \mid Ax = 0\}$ $\subseteq \mathbb{R}^n$
- Range = linear span of the columns of A
- Rank of a matrix $\mathrm{rank}(A) = \dim(\mathrm{Ran}(A))$ $\leq n$
- $\mathrm{Ran}(A) \subseteq \mathbb{R}^m \rightarrow$ rank $(A) \leq m \rightarrow$

$$\text{rank } (A) \leq \min\{m, n\}$$

- rank (A) = number of linearly independent columns of A = number of linearly independent rows of A
- A is of full rank if $\mathrm{rank}(A) = \min\{m, n\}$. Otherwise it is rank-deficient.

Rank+Nullity theorem for an $m \times n$ matrix:

$$dim(Ran(A)) + dim(Null(A)) = n$$

Apply to A^T: $dim(Ran(A^T)) + dim(Null(A^T)) = m \rightarrow$

$$\text{rank}(A) + dim(Null(A^T)) = m$$

➤ Terminology:

- $dim(Null(A))$ is the Nullity of A [Another term: co-rank]

✍15 Show that $A \in \mathbb{R}^{m \times n}$ is of rank one iff [if and only if] there exist two nonzero vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$ such that $A = uv^T$. What are the eigenvalues and eigenvectors of A?

✍16 Is it true that: $\textbf{rank}(A) = \textbf{rank}(\bar{A}) = \textbf{rank}(A^T) = \textbf{rank}(A^H)$?

✍17 Matlab exercise: explore the matlab function `rank`.

✍18 Matlab exercise: explore the matlab function `rref`.

➤ No `rref` function in numpy – [see sympy]

✍19 Find the range and null space of the following matrix: Verify your result with matlab [hint: use `null`, `rank`, `rref`]

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & 2 & 3 \\ 1 & -2 & -1 \\ 2 & -1 & 1 \end{pmatrix}$$

Square matrices, matrix inversion, eigenvalues

- Square matrix: $n = m$, i.e., $A \in \mathbb{R}^{n\times n}$
- Identity matrix: square matrix with

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$$

- Notation: I.
- Property: $AI = IA = A$
- Inverse of A (when it exists) is a matrix C such that

$$AC = CA = I$$

Notation: A^{-1} .

Eigenvalues and eigenvectors

A complex scalar λ is called an eigenvalue of a square matrix A if there exists a nonzero vector u in $\mathbb{C}^n$ such that $Au = \lambda u$. The vector u is called an eigenvector of A associated with λ. The set of all eigenvalues of A is the 'spectrum' of A. Notation: $\Lambda(A)$.

➤ λ is an eigenvalue iff the columns of $A - \lambda I$ are linearly dependent.

➤ ... equivalent to saying that its rows are linearly dependent. So: there is a nonzero vector w such that

$$w^H(A - \lambda I) = 0$$

➤ w is a left eigenvector of A (u= right eigenvector)

➤ λ is an eigenvalue iff $\boxed{\det(A - \lambda I) = 0}$

Eigenvalues/vectors

➤ An eigenvalue is a root of the Characteristic polynomial:

$$p_A(\lambda) = \det(A - \lambda I)$$

➤ So there are n eigenvalues (counted with their multiplicities).

➤ The multiplicity of these eigenvalues as roots of p_A are called algebraic multiplicities.

➤ The geometric multiplicity of an eigenvalue λ_i is the number of linearly independent eigenvectors associated with λ_i.

➤ Geometric multiplicity is $\leq$ algebraic multiplicity.

➤ An eigenvalue is simple if its (algebraic) multiplicity is one. It is semi-simple if its geometric and algebraic multiplicities are equal.

➤ Two matrices A and B are similar if there exists a nonsingular matrix X such that

$$A = XBX^{-1}$$

✍20 Eigenvalues of A and B are the same. What about eigenvectors?

➤ Note: A and B represent the same mapping using 2 different bases.

Fundamental Problem: Given A, find X so that B has a simpler structure (e.g., diagonal) $\rightarrow$ Eigenvalues of B easier to compute

Definition: A is diagonalizable if it is similar to a diagonal matrix

➤ We will revisit these notions later in the semester

✍21 Given a polynomial $p(t)$ how would you define $p(A)$?

✍22 Given a function $f(t)$ (e.g., e^t) how would you define $f(A)$? [Leave the full justification for next chapter]

✍23 If A is nonsingular what are the eigenvalues/eigenvectors of A^{-1}?

✍24 What are the eigenvalues/eigenvectors of A^k for a given integer power k?

✍25 What are the eigenvalues/eigenvectors of $p(A)$ for a polynomial p?

✍26 What are the eigenvalues/eigenvectors of $f(A)$ for a function f? [Diagonalizable case]

✍27 For two $n \times n$ matrices A and B are the eigenvalues of AB and BA the same?

✍28 Review the Jordan canonical form; see short description in sec. 1.8.2 of:

http://www.cs.umn.edu/~saad/eig_book_2ndEd.pdf

Define the eigenvalues, and eigenvectors from the Jordan form.

➤ Spectral radius = The maximum modulus of the eigenvalues

$$\rho(A) = \max_{\lambda \in \lambda(A)} |\lambda|.$$

➤ Trace of A = sum of diagonal elements of A.

$$\text{Tr}(A) = \sum_{i=1}^{n} a_{ii}.$$

Properties:

1 *Tr*(A) = sum of the eigenvalues of A counted with their multiplicities.

2 $\det(A)$ = product of the eigenvalues of A counted with their multiplicities.

✍29 Trace, spectral radius, and determinant of

$$A = \begin{pmatrix} 2 & 1 \\ 3 & 0 \end{pmatrix}.$$

Review of Determinants: summary of main results

➤ [For review only – will *not* be covered in detail in class]

➤ A determinant of an $n \times n$ matrix is a number associated with this matrix. Its definition is complex for the general case $\rightarrow$ We start with $n = 2$ and list important properties for this case.

- Determinant of a 2×2 matrix is:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$$

- Notation : $\det(A)$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

➤ Next we list the main properties of determinants. Some of these properties can be extended to the $n \times n$ case to be defined later.

➤ Properties written for columns (easier to write) but are also true for rows

Notation: We let $\boldsymbol{A} = [\boldsymbol{u}, \boldsymbol{v}]$ columns $\boldsymbol{u}$, and $\boldsymbol{v}$ are in $\mathbb{R}^2$.

1 If $\boldsymbol{v} = \alpha \boldsymbol{u}$ then $\det(\boldsymbol{A}) = \boldsymbol{0}$.

➤ Determinant of linearly dependent vectors is zero

➤ If any one column is zero then determinant is zero

2 Interchanging columns or rows: $\det[\boldsymbol{v}, \boldsymbol{u}] = -\det[\boldsymbol{u}, \boldsymbol{v}]$

3 Linearity: $\det[\boldsymbol{u}, \alpha \boldsymbol{v} + \beta \boldsymbol{w}] = \alpha \det[\boldsymbol{u}, \boldsymbol{v}] + \beta \det[\boldsymbol{u}, \boldsymbol{w}]$

➤ $\det(\boldsymbol{A})$ = linear function of each column (individually)

➤ $\det(\boldsymbol{A})$ = linear function of each row (individually)

✍30 What is the determinant $\det[\boldsymbol{u}, \boldsymbol{v} + \alpha \boldsymbol{u}]$?

4 Determinant of transpose $\det(A) = \det(A^T)$

5 Determinant of Identity $\det(I) = 1$

➤ Notation: Diagonal matrix $D = \text{Diag}\{d_1, d_2, \cdots, d_n\}$ with $D_{ii} = d_i, i = 1 : n$

6 Determinant of a diagonal matrix: $\det(D) = d_1 d_2 \cdots d_n$

7 Determinant of a triangular matrix (upper or lower) $\det(T) = a_{11} a_{22} \cdots a_{nn}$

8 Determinant of product of matrices [IMPORTANT] $\det(AB) = \det(A)\det(B)$

9 Consequence: Determinant of inverse $\det(A^{-1}) = \frac{1}{\det(A)}$

✍31 What is the determinant of αA (for 2×2 matrices)?

✍32 What can you say about the determinant of a matrix that satisfies $A^2 = I$?

✍33 Is it true that $\det(A + B) = \det(A) + \det(B)$?

Determinants – 3×3 case

- We will define 3×3 determinants from 2×2 determinants:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- This is an expansion of the det. with respect to its 1st row.
- Note the alternating sign in expansion.

1st term $= a_{11} \times$ det of matrix obtained by deleting 1st row and 1st column.

2nd term $= -a_{12} \times$ det of matrix obtained by deleting row 1 and column 2.

3rd term $= a_{13} \times$ det of matrix obtained by deleting row 1 and column 3.

✍34 Calculate $\begin{vmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \\ -1 & 2 & 1 \end{vmatrix}$

➤ We will now generalize this definition to any dimension recursively. Need to define following notation.

We denote by A_{ij} the $(n-1) \times (n-1)$ matrix obtained by deleting row i and column j from A.

Example: If $A = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 2 & -1 \\ -1 & 2 & 1 \end{bmatrix}$ Then: $A_{11} = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$;

$$A_{12} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}; \; A_{13} = \begin{bmatrix} 1 & 2 \\ -1 & 2 \end{bmatrix}; \; A_{23} = \begin{bmatrix} 2 & 3 \\ -1 & 2 \end{bmatrix}$$

Definition The determinant of a matrix $A = [a_{ij}]$ is the sum

$$\det(A) = + a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - a_{14}\det(A_{14}) + \cdots + (-1)^{1+n} a_{1n}\det(A_{1n})$$

- Note the alternating signs

- We can write this as :

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} a_{1j} \det(A_{1j})$$

- which is an expansion with respect to the 1st row.

✍35 Let A be a nonsingular diagonal $n \times n$ matrix. Show that:

$$\log \det(A) = \text{Trace}(\log(A))$$

Generalization: Cofactors

Define $c_{ij} = (-1)^{i+j} \det A_{ij}$ = cofactor of entry i, j

➤ Then $\det(A)$ can be expanded with respect to i-th row as follows:

$$\det(A) = a_{i1}c_{i1} + a_{i2}c_{i2} + \cdots + a_{in}c_{in}$$

➤ Note i is fixed. Can be done for any i [same result each time]. Case $i = 1$ corresponds to definition given earlier

➤ Similar expressions for expanding w.r.t. column j (now j is fixed)

✍36 Let $C = \{c_{ij}\}_{i,j=1:n} \equiv$ matrix of cofactors. Show that $AC^T = \det(A) \times I$. So $A^{-1} = ?$