✍1 Consider

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues of A? their algebraic multiplicities? their geometric multiplicities? Is one a semi-simple eigenvalue?

Solution: The eigenvalues of A are 1, and 2. The algebraic multiplicity of 1 is 2. To get the geometric multiplicity of the eigenvalue $\lambda = 1$ we need to eigenvectors. For this we need to solve:

$$\begin{pmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} u = 0.$$

There is only one solution vector (up to a product by a scalar) namely:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So the geometric multiplicity is one. $\square$

✍2 Same questions if a_{33} is replaced by one.

Solution: The matrix become

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and now we have one eigenvalue algebraic multiplicity 3.

To get the geometric multiplicity of the eigenvalue $\lambda = 1$ we need to eigenvectors. For this we need to solve:

$$\begin{pmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} u = 0.$$

we still get a geometric mult. of 1. $\square$

✍3 Same questions if in addition a_{12} is replaced by zero.

Solution: Solution: The matrix become

$$\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and we also have one eigenvalue with algebraic multiplicity 3. The geometric multiplicity increases to 2. □

✍4 Show that there is at least one eigenvalue and eigenvector of $\boldsymbol{A}$: $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{\lambda}\boldsymbol{x}$, with $\|\boldsymbol{x}\|_2 = 1$

Solution: This comes from the fact that the equation $\boldsymbol{P}_{A}(\boldsymbol{\lambda}) = \det(\boldsymbol{A} - \boldsymbol{\lambda}\boldsymbol{I}) = 0$ is a polynomial equation and as such it must have at least one root - a well-known result. □

✍5 There is a unitary transformation $\boldsymbol{P}$ such that $\boldsymbol{P}\boldsymbol{x} = \boldsymbol{e}_1$. How do you define $\boldsymbol{P}$?

Solution: This is just the Householder transform.. See Lecture notes set number 8. □

✍6 Show that $PAP^H = \left(\begin{array}{c|c} \lambda & ** \\ \hline 0 & A_2 \end{array}\right)$.

Solution: This is equivalent to showing that $PAP^H e_1 = \lambda e_1$. We have

$$PAP^H e_1 = PAPe_1 = P(Ax) = P(\lambda x) = \lambda Px = \lambda e_1$$

□

✍9 Another proof altogether: use Jordan form of A and QR factorization **Solution:** Jordan form:

$$A = XJX^{-1}$$

Let $X = QR_0$ then:

$$A = QR_0 J R_0^{-1} Q^H \equiv QRQ^H \quad \text{with} \quad R = R_0 J R_0^{-1}$$

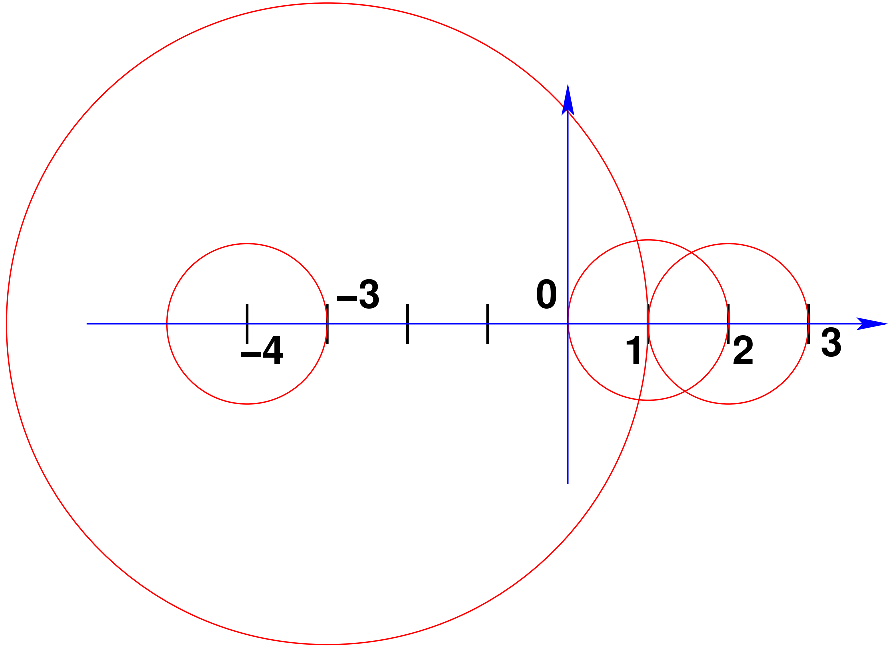
□

✍10 Find a region of the complex plane where the eigenvalues of the following matrix are located:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & -2 & -3 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & -4 \end{pmatrix}$$

Solution: Use Gershgorin's theorem. There are 4 disks:

$$D_1 = D(1,1); \qquad D_2 = D(2,1)$$
$$D_3 = D(-3,4); \qquad D_4 = D(-4,1)$$



The last disk is included in the 3rd. The spectrum is included in the union of the 3 other disks. ☐

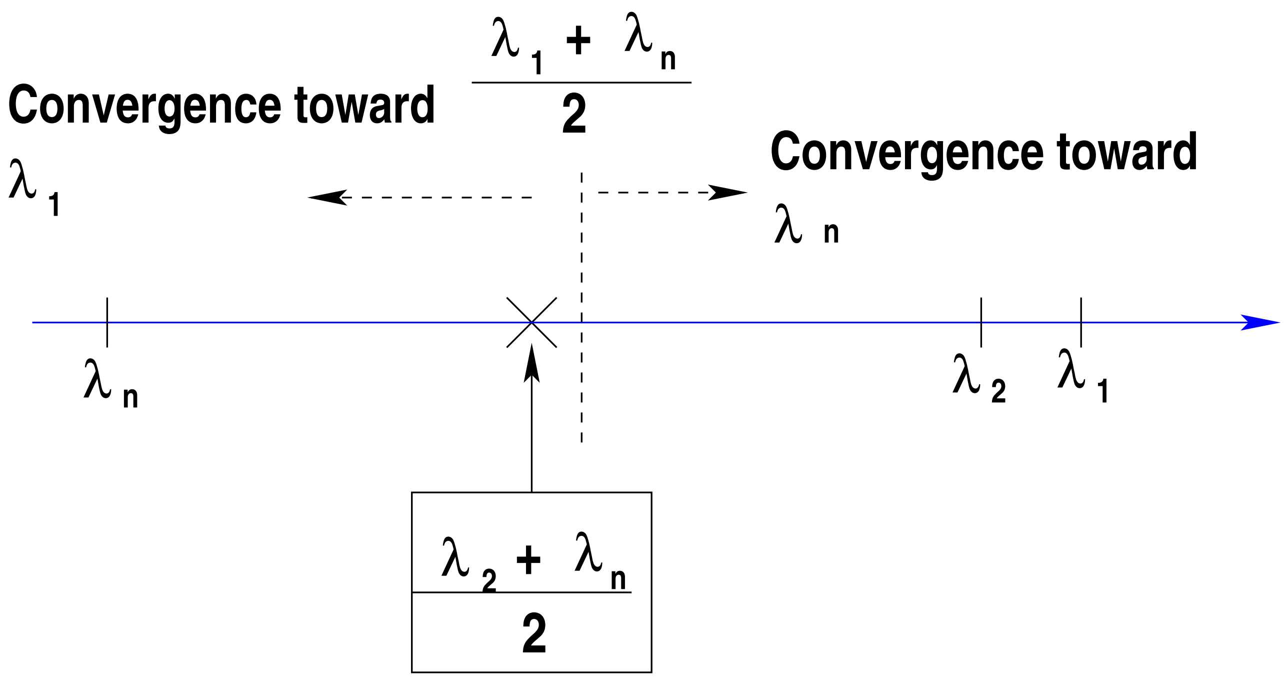
✍11 Convergence factor $\phi(\sigma)$ as a function of σ.

Solution: The eigenvalues of the shifted matrix are $\lambda_i - \sigma$. When $\sigma > (\lambda_1 + \lambda_n)/2$ then the algorithm will converge toward λ_n because $|\lambda_n - \sigma| > |\lambda_1 - \sigma|$. We will ignore this case.

Assume now that $\sigma < (\lambda_1 + \lambda_n)/2$. If $\sigma < (\lambda_2 + \lambda_n)/2$ then largest eigenvalue of $A - \sigma$ is $\lambda_1 - \sigma$

and second largest is $\lambda_2 - \sigma$. If $\sigma \geq (\lambda_2 + \lambda_n)/2$ then largest eigenvalue of $A - \sigma$ is $\lambda_n - \sigma$ and second largest is $\lambda_2 - \sigma$. Therefore, setting $\mu = (\lambda_2 + \lambda_n)/2$, we get

$$\phi(\sigma) = \begin{cases} \frac{|\lambda_2 - \sigma|}{|\lambda_1 - \sigma|} = \frac{\lambda_2 - \sigma}{\lambda_1 - \sigma} & \text{if } \sigma < \mu \\ \frac{|\lambda_n - \sigma|}{|\lambda_1 - \sigma|} = \frac{\sigma - \lambda_n}{\lambda_1 - \sigma} & \text{if } \sigma > \mu \end{cases}$$



Note that for $\sigma < \mu$ we have $\phi(\sigma) = 1 - (\lambda_1 - \lambda_2)/(\lambda_1 - \sigma)$ which is a decreasing function while when $\sigma > \mu$ we have $\phi(\sigma) = -1 + (\lambda_1 - \lambda_n)/(\lambda_1 - \sigma)$ which is an increasing function. The min. is reached when these 2 values are equal which leads to the solution $\sigma_{opt} = (\lambda_n + \lambda_2)/2$ $\square$

Additional notes.

In discussing Gerschgorin theorem it was stated:

- Refinement: if disks are all disjoint then each of them contains one eigenvalue

Question: Why?

Solution:

Consider the matrix $A(t) = D + t(A - D)$ where D is the diagonal of A. Note $A(0) = D$, $A(1) = A$. Consider the n disks as t varies from $t = 0$ to $t = 1$. When $t = 0$ each disk contains exactly one eigenvalue. As t increases (in a continuous way) fom 0 to one – each disk will still contain one eigenvalue

- by a continuity argument [you cannot have an eigenvalue jump suddently - from one disk to another- this would be a dicontinuous behavior]. The argument can be adapted to the case where two disks touch each other at one point (only): it is now possible to have two eigenvalues at the intersection of the disks - coming from each of the t2o disks.