✍2 Use the min-max theorem to show that $\|A\|_2 = \sigma_1(A)$ - the largest singular value of A.

Solution: This comes from the fact that:

$$\begin{aligned} \|A\|_2^2 &= \max_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} \\ &= \max_{x\neq 0} \frac{(Ax, Ax)}{(x, x)} \\ &= \max_{x\neq 0} \frac{(A^T Ax, A)}{(x, x)} \\ &= \lambda_{max}(A^T A) \\ &= \sigma_1^2 \end{aligned}$$

□

✍3 Suppose that $A = LDL^T$ where L is unit lower triangular, and D diagonal. How many negative eigenvalues does A have?

Solution: It has as many negative eigenvalues as there are negative entries in D ☐

✍4 Assume that A is tridiagonal. How many operations are required to determine the number of negative eigenvalues of A?

Solution: The rough answer is $O(n)$ – because an LU (and therefore LDLT) factorization costs $O(n)$. Based on doing the LU factorization of a triagonal matrix, a more accurate answer is $3n$ operations. ☐

✍5 Devise an algorithm based on the inertia theorem to compute the i-th eigenvalue of a tridiagonal matrix.

Solution: Here is a matlab script:

```
   function [sigma] = bisect(d, b, i, tol)
%% function [sigma] = bisect(d, b, i, tol)
%% d   = diagonal of T
```

```
%% b   = co-diagonal
%% i   = compute i-th eigenvalue
%% tol = tolerance used for stopping
     b(1) = 0;
   n = length(d);
 %%-------------------- guershgorin
   tmin = d(n) - abs(b(n));
   tmax = d(n) + abs(b(n));
   for j=1:n-1
     rho = abs(b(j)) + abs(b(j+1));
     tmin = min(tmin, d(j)-rho);
     tmax = max(tmax, d(j)+rho);
   end
   tol = tol*(tmax-tmin);
   for iter=1:100
     sigma = 0.5*(tmin+tmax);
     count = sturm(d, b, sigma);
     if (count >= i)
       tmin = sigma;
     else
       tmax = sigma;
     end
     if (tmax - tmin) < tol
       break
     end
   end
```

□

✍6 What is the inertia of the matrix

$$\begin{pmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \boldsymbol{F}^T & 0 \end{pmatrix}$$

where $\boldsymbol{F}$ is $\boldsymbol{m} \times \boldsymbol{n}$, with $\boldsymbol{n} < \boldsymbol{m}$, and of full rank?

[Hint: use a block LU factorization]

Solution: We start with

$$\begin{aligned}
\begin{pmatrix} \boldsymbol{I} & \boldsymbol{F} \\ \boldsymbol{F}^T & 0 \end{pmatrix} &= \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{F}^T & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{F} \\ 0 & -\boldsymbol{F}^T\boldsymbol{F} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{F}^T & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{F}^T\boldsymbol{F} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & \boldsymbol{F} \\ 0 & \boldsymbol{I} \end{pmatrix} \\
&= \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{F}^T & \boldsymbol{I} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & 0 \\ 0 & -\boldsymbol{F}^T\boldsymbol{F} \end{pmatrix} \begin{pmatrix} \boldsymbol{I} & 0 \\ \boldsymbol{F}^T & \boldsymbol{I} \end{pmatrix}^T
\end{aligned}$$

This is of the form XDX^T where X is invertible.. Therefore the inertia is the same as that of the block diagonal matrix which is: m positive eigenvalues (block I) and n negative eigenvalues since $-F^TF$ is $n \times n$ and negative definite.

✍7 Let $\|A_O\|_I = \max_{i\neq j} |a_{ij}|$. Show that

$$\|A_O\|_F \leq \sqrt{n(n-1)}\|A_O\|_I$$

Solution: This is straightforward:

$$\|A_O\|_F^2 = \sum_{i\neq j} |a_{ij}|^2 \leq n(n-1) \max_{i\neq j} |a_{ij}|^2 = n(n-1)\|A_O\|_I^2.$$

□

✍8 Use this to show convergence in the case when largest entry is zeroed at each step.

Solution: If we call B_k the matrix A_O after each rotation then we have according to result in the previous page and using the previous exercise:

$$\begin{aligned}
\|B_{k+1}\|_F^2 &= \|B_k\|_F^2 - 2a_{pq}^2 \\
&= \|B_k\|_F^2 - 2\|B_k\|_I^2 \\
&\leq \|B_k\|_F^2 - \frac{2}{n(n+1)}\|B_k\|_F^2 \\
&= \left[1 - \frac{2}{n(n+1)}\right] \|B_k\|_F^2
\end{aligned}$$

which shows that the norm will be decreasing by factor less than a constant that is less than one - therefore it converges to zero. □