

 1 Unitary matrices preserve the 2-norm.

**Solution:** The proof takes only one line if we use the result  $(Ax, y) = (x, A^H y)$ :

$$\|Qx\|_2^2 = (Qx, Qx) = (x, Q^H Qx) = (x, x) = \|x\|_2^2. \quad \square$$

 3 When do we have equality in Cauchy-Schwarz?

**Solution:** From the proof of Cauchy-Schwarz it can be seen that we have equality when  $x = \lambda y$ , i.e., when they are colinear.  $\square$

 4 Expand  $(x + y, x + y)$  – What does Cauchy-Schwarz imply?

**Solution:** You will see that you can derive the triangle inequality from this expansion and the Cauchy-

Schwarz inequality.  $\square$ .

- Proof of the Hölder inequality.

$$|(\mathbf{x}, \mathbf{y})| \leq \|\mathbf{x}\|_p \|\mathbf{y}\|_q, \text{ with } \frac{1}{p} + \frac{1}{q} = 1$$

Proof: For any  $z_i, v_i$  all nonnegative we have, setting  $\zeta = \sum z_i$ ,

$$\begin{aligned} \left( \sum (z_i/\zeta) v_i \right)^p &\leq \sum (z_i/\zeta) v_i^p \text{ (convexity)} \rightarrow \\ \left( \sum z_i v_i \right)^p &\leq \left[ \sum (z_i/\zeta) v_i^p \right] \zeta^p = \left[ \sum z_i v_i^p \right] \zeta^{p-1} \rightarrow \\ \sum z_i v_i &\leq \left[ \sum z_i v_i^p \right]^{1/p} \zeta^{(p-1)/p} \\ \sum z_i v_i &\leq \left[ \sum z_i v_i^p \right]^{1/p} \left[ \sum z_i \right]^{1/q} \end{aligned}$$

Now take  $z_i = x_i^q$ , and  $v_i = y_i * x_i^{1-q}$ . Then  $z_i v_i = x_i y_i$  and:

$$z_i v_i^p = x_i^q * (y_i * x_i^{1-q})^p = y_i^p * x_i^{q+p-pq} = y_i^p * x_i^0 = y_i^p \quad \square$$

 5 Second triangle inequality.

**Solution:** Start by invoking the triangle inequality to write:

$$\|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\| \rightarrow \|x\| - \|y\| \leq \|x - y\|$$

Next exchange the roles of  $x$  and  $y$ :

$$\|y\| - \|x\| \leq \|y - x\| = \|x - y\|$$

The two inequalities  $\|x\| - \|y\| \leq \|x - y\|$  and  $\|y\| - \|x\| \leq \|x - y\|$  yield the result since they imply that

$$-\|x - y\| \leq \|x\| - \|y\| \leq \|x - y\|$$



 6 Consider the metric  $d(x, y) = \max_i |x_i - y_i|$ . Show that any norm in  $\mathbb{R}^n$  is a continuous function

with respect to this metric.

**Solution:** We need to show that we can make  $\|\mathbf{y}\|$  arbitrarily close to  $\|\mathbf{x}\|$  by making  $\mathbf{y}$  ‘close’ enough to  $\mathbf{x}$ , where ‘close’ is measured in terms of the infinity norm distance  $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_\infty$ . Define  $\mathbf{u} = \mathbf{x} - \mathbf{y}$  and write  $\mathbf{u}$  in the canonical basis as  $\mathbf{u} = \sum_{i=1}^n \delta_i \mathbf{e}_i$ . Then:

$$\|\mathbf{u}\| = \left\| \sum_{i=1}^n \delta_i \mathbf{e}_i \right\| \leq \sum_{i=1}^n |\delta_i| \|\mathbf{e}_i\| \leq \max |\delta_i| \sum_{i=1}^n \|\mathbf{e}_i\|$$

Setting  $M = \sum_{i=1}^n \|\mathbf{e}_i\|$  we get  $\boxed{\|\mathbf{u}\| \leq M \max |\delta_i| = M \|\mathbf{x} - \mathbf{y}\|_\infty}$

Let  $\epsilon$  be given and take  $\mathbf{x}, \mathbf{y}$  such that  $\|\mathbf{x} - \mathbf{y}\|_\infty \leq \frac{\epsilon}{M}$ . Then, by using the second triangle inequality we obtain:

$$| \|\mathbf{x}\| - \|\mathbf{y}\| | \leq \|\mathbf{x} - \mathbf{y}\| \leq M \max \delta_i \leq M \frac{\epsilon}{M} = \epsilon.$$

This means that we can make  $\|\mathbf{y}\|$  arbitrarily close to  $\|\mathbf{x}\|$  by making  $\mathbf{y}$  close enough to  $\mathbf{x}$  in the sense of the defined metric. Therefore  $\|\cdot\|$  is continuous.  $\square$

 7 In  $\mathbb{R}^n$  (or  $\mathbb{C}^n$ ) all norms are equivalent.

**Solution:** We will do it for  $\phi_1 = \|\cdot\|$  some norm, and  $\phi_2 = \|\cdot\|_\infty$  [and one can see that all other cases will follow from this one].

1. Need to show that for some  $\alpha$  we have  $\|\mathbf{x}\| \leq \alpha\|\mathbf{x}\|_\infty$ . Express  $\mathbf{x}$  in the canonical basis of  $\mathbb{R}^n$  as  $\mathbf{x} = \sum x_i \mathbf{e}_i$  [look up canonical basis  $\mathbf{e}_i$  from your csci2033 class.] Then

$$\|\mathbf{x}\| = \left\| \sum x_i \mathbf{e}_i \right\| \leq \sum |x_i| \|\mathbf{e}_i\| \leq \max |x_i| \sum \|\mathbf{e}_i\| = \|\mathbf{x}\|_\infty \alpha$$

where  $\alpha = \sum \|\mathbf{e}_i\|$ .

2. We need to show that there is a  $\beta$  such that  $\|\mathbf{x}\| \geq \beta\|\mathbf{x}\|_\infty$ . Assume  $\mathbf{x} \neq \mathbf{0}$  and consider  $\mathbf{u} = \mathbf{x}/\|\mathbf{x}\|_\infty$ . Note that  $\mathbf{u}$  has infinity norm equal to one. Therefore it belongs to the closed and bounded set  $S_\infty = \{\mathbf{v} \mid \|\mathbf{v}\|_\infty = 1\}$ . Since norms are continuous (seen earlier), the minimum of the norm  $\|\mathbf{u}\|$  for all

$u$ 's in  $S_\infty$  is reached, i.e., there is a  $u_0 \in S_\infty$  such that

$$\min_{u \in S_\infty} \|u\| = \|u_0\|.$$

Let us call  $\beta$  this minimum value, i.e.,  $\|u_0\| = \beta$ . Note in passing that  $\beta$  cannot be equal to zero otherwise  $u_0 = 0$  which would contradict the fact that  $u_0$  belongs to  $S_\infty$  [all vectors in  $S_\infty$  have infinity norm equal to one.] The result follows because  $u = x/\|x\|_\infty$ , and so, remembering that  $u = x/\|x\|_\infty$ , we obtain

$$\left\| \frac{x}{\|x\|_\infty} \right\| \geq \beta \rightarrow \|x\| \geq \beta \|x\|_\infty$$

This completes the proof  $\square$

**Ex 8** Show that for any  $x$ :  $\frac{1}{\sqrt{n}} \|x\|_1 \leq \|x\|_2 \leq \|x\|_1$

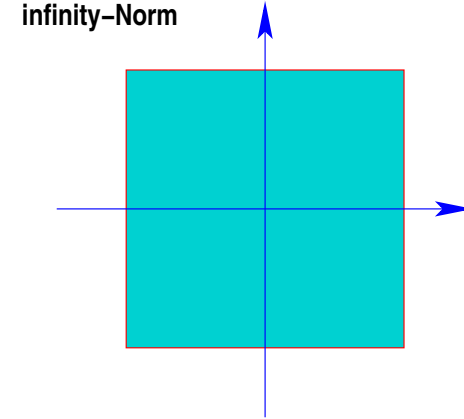
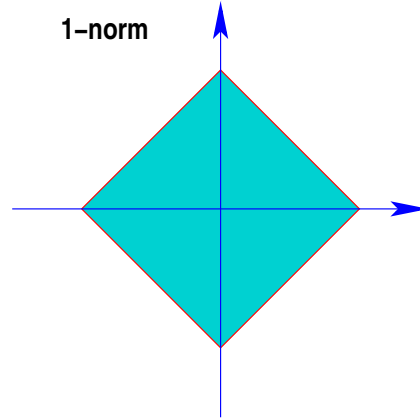
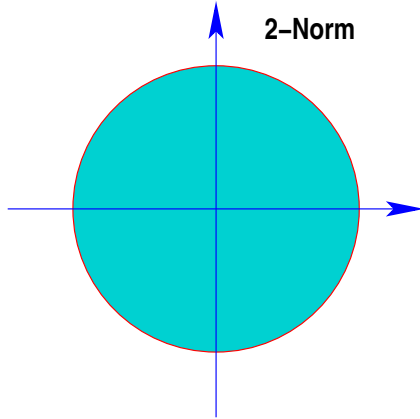
**Solution:** For the right inequality, it is easy to see that  $\|x\|_2 \leq \|x\|_1$  because  $\sum_i x_i^2 \leq [\sum_i |x_i|]^2$

For the left inequality, we rely on Cauchy-Schwarz. If we call  $\mathbf{1}$  the vector of all ones, then:

$$\|x\|_1 = \sum_i |x_i| \cdot \mathbf{1} \leq \|x\|_2 \|\mathbf{One}\|_2 = \sqrt{n} \|x\|_2$$



 9 Unit balls in  $\mathbb{R}^2$ .



 14 Show that  $\rho(A) \leq \|A\|$  for any matrix norm.

**Solution:** Let  $\lambda$  be the largest (in modulus) eigenvalue of  $A$  with associated eigenvector  $u$ . Then

$$Au = \lambda u \rightarrow \frac{\|Au\|}{\|u\|} = |\lambda| = \rho(A)$$

This implies that

$$\rho(A) \leq \max_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \|A\|$$



**15** Is  $\rho(A)$  a norm?

**Solution:** This was answered in the notes.

**16** Given a function  $f(t)$  (e.g.,  $e^t$ ) how would you define  $f(A)$ ? [You may limit yourself to the case when  $A$  is diagonalizable]



**Solution:** The easiest way would be through the Taylor series expansion..

$$f(A) = f(0)I + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 \dots \frac{f^{(k)}(0)}{k!}A^k + \dots$$

However, this will require a justification: Will this expression ‘converge’ as the number of terms goes to infinity? This is where norms are useful.

In the simplest case where  $A$  is diagonalizable you can write  $A = XDX^{-1}$  and then consider the  $k$ -term part of the Taylor series expression above:

$$\begin{aligned} F_k &= f(0)I + \frac{f'(0)}{1!}A + \frac{f''(0)}{2!}A^2 + \dots + \frac{f^{(k)}(0)}{k!}A^k \\ &= X \left[ f(0)I + \frac{f'(0)}{1!}D + \frac{f''(0)}{2!}D^2 + \dots + \frac{f^{(k)}(0)}{k!}D^k \right] X^{-1} \\ &\equiv XD_kX^{-1} \end{aligned}$$

where  $D_k$  is the matrix inside the brackets in line 2 of above equations. The  $i - th$  diagonal entry of  $D_k$

is of the form

$$f_k(\lambda_i) = f(0) + \frac{f'(0)}{1!}\lambda_i + \frac{f''(0)}{2!}\lambda_i^2 + \dots + \frac{f^{(k)}(0)}{k!}\lambda_i^k,$$

which is just the  $k$ -term part of the Taylor series expansion of  $f(\lambda_i)$ . Each of these will converge to  $f(\lambda_i)$ .

Now it is easy to complete the argument. If we call  $D_f$  the diagonal matrix whose  $i$ th diagonal entry is  $f(\lambda_i)$  and  $f_A$  the matrix defined by

$$f_A = X D_f X^{-1},$$

then clearly

$$\begin{aligned} \|F_k - F_A\|_2 &= \|X(D_k - D_A)X^{-1}\|_2 \leq \|X\|_2 \|X^{-1}\|_2 \|D_k - D_A\|_2 \\ &\leq \|X\|_2 \|X^{-1}\|_2 \max_i |f_k(\lambda_i) - f(\lambda_i)| \end{aligned}$$

which converges to zero as  $k$  goes to infinity.  $\square$

**17** The eigenvalues of  $A^H A$  and  $A A^H$  are real nonnegative.

**Solution:** Let us show it for  $A^H A$  [the other case is similar] If  $\lambda, u$  is an eigenpair of  $A^H A$  then  $(A^H A)u =$

$\lambda u$ . Take inner products with  $u$  on both sides. Then:

$$\lambda(u, u) = ((A^H A)u, u) = (Au, Au) = \|Au\|^2$$

Therefore,  $\lambda = \|Au\|^2/\|u\|^2$  which is a real nonnegative number.  $\square$

[Note: 1) Observe how simple the proof is for such an important fact. It is based on the result  $(Ax, y) = (x, A^H y)$ . 2) The singular values of  $A$  are the square roots of the eigenvalues of  $A^H A$  if  $m \geq n$  or those of the eigenvalues of  $AA^H$  if  $m < n$ . So there are always  $\min(m, n)$  singular values. This is really just a preliminary definition as we need to refer to singular values often – but we will see singular values and the singular value decomposition in great detail later.]

 18 Prove that when  $A = uv^T$  then  $\|A\|_2 = \|u\|_2 \|v\|_2$ .

**Solution:** We start by dealing with the eigenvalues of an arbitrary matrix of the form  $A = uv^T$  where both

$u$  and  $v$  are in  $\mathbb{R}^n$ . From  $Ax = \lambda x$  we get:

$$uv^T x = \lambda x \rightarrow (v^T x)u = \lambda x$$

Notice that we did this because  $v^T x$  is a scalar. We have 2 cases.

Case 1:  $v^T x = 0$ . In this case it is clear that the equation  $Ax = \lambda x$  is satisfied with  $\lambda = 0$ . So any vector that is orthogonal to  $v$  is an eigenvector of  $A$  associated with the eigenvalue  $\lambda = 0$ . (It can be shown that the eigenvalue 0 is of multiplicity  $n - 1$ ).

Case 2:  $v^T x \neq 0$ . In this case it is clear that the equation  $Ax = \lambda x$  is satisfied with  $\lambda = v^T u$  and  $x = u$ . So  $u$  is an eigenvector of  $A$  associated with the eigenvalue  $v^T u$ .

In summary the matrix  $uv^T$  has only two eigenvalues: 0, and  $v^T u$ .

Going back to the original question, we consider now  $A = uv^T$  and we are interested in the 2-norm of  $A$ .

We have

$$\|A\|_2^2 = \rho(A^T A) = \rho(vu^T uv^T) = \|u\|_2^2 \rho(vv^T) = \|u\|_2^2 \|v\|_2^2.$$

The last relation comes from what was done above to determine the eigenvalues of  $vv^T$ . So in the end,

$$\|A\|_2 = \|u\|_2 \|v\|_2. \quad \square$$

19 In this case what is  $\|A\|_F$ ?

**Solution:** Only the last part of the above answer changes ( $\rho$  is replaced by  $\text{Tr}$ ) and you will find that actually the Frobenius norm of  $uv^T$  is again equal to  $\|u\|_2 \|v\|_2$ .  $\square$

**Proof of Cauchy-Schwarz inequality:**

$$|(x, y)|^2 \leq (x, x) (y, y). \quad (1)$$

**Proof:** We begin by expanding  $(x - \lambda y, x - \lambda y)$  using properties of inner products:

$$(x - \lambda y, x - \lambda y) = (x, x) - \bar{\lambda}(x, y) - \lambda(y, x) + |\lambda|^2(y, y).$$

If  $y = 0$  then the inequality is trivially satisfied. Assume that  $y \neq 0$  and take  $\lambda = (x, y)/(y, y)$ . Then, from the above equality,  $(x - \lambda y, x - \lambda y) \geq 0$  shows that

$$\begin{aligned} 0 \leq (x - \lambda y, x - \lambda y) &= (x, x) - 2\frac{|(x, y)|^2}{(y, y)} + \frac{|(x, y)|^2}{(y, y)} \\ &= (x, x) - \frac{|(x, y)|^2}{(y, y)}, \end{aligned}$$

which yields the result.  $\square$