✍1 Show that $\kappa(I) = 1$;

Solution: This is obvious because for any matrix norm $\|I\| = \|I^{-1}\| = 1$. $\square$

✍2 Show that $\kappa(A) \geq 1$;

Solution: We have $\|AA^{-1}\| = \|I\| = 1$ therefore $1 = \|AA^{-1}\| \leq \|A\| \, \|A^{-1}\| = \kappa(A)$ $\square$

✍5 Show that if $\|E\|/\|A\| \leq \delta$ and $\|e_b\|/\|b\| \leq \delta$ then

$$\frac{\|x - y\|}{\|x\|} \leq \frac{2\delta\kappa(A)}{1 - \delta\kappa(A)}$$

Solution: From the main theorem (theorem 1) we have

$$\frac{\|x-y\|}{\|x\|} \le \frac{\|A^{-1}\|\,\|A\|}{1-\|A^{-1}\|\,\|E\|}\left(\frac{\|E\|}{\|A\|}+\frac{\|e_b\|}{\|b\|}\right)$$

If $\|E\| \le \delta$ and $\|e_b\|/\|b\| \le \delta$ then:

$$\begin{aligned}\frac{\|x-y\|}{\|x\|} &\le \frac{\kappa(A)\times 2\delta}{1-\|A^{-1}\|\,\|E\|}\\ &\le \frac{2\delta\kappa(A)}{1-\|A^{-1}\|\|A\|\times(\|E\|/\|A\|)}\\ &\le \frac{2\delta\kappa(A)}{1-\delta\kappa(A)}.\end{aligned}$$

$\square$

✍9 Show that $\frac{\|x-\tilde{x}\|}{\|x\|} \ge \frac{1}{\kappa(A)}\frac{\|r\|}{\|b\|}$.

Solution: As before we start with noting that $A(x-\tilde{x}) = b - A\tilde{x} = r$. So:

$$\|r\| \le \|A\|\|x-\tilde{x}\| \to \frac{\|r\|}{\|b\|} \le \|A\|\frac{\|x-\tilde{x}\|}{\|b\|}$$

Next from $\|\boldsymbol{x}\| = \|\boldsymbol{A}^{-1}\boldsymbol{b}\| \leq \|\boldsymbol{A}^{-1}\|\|\boldsymbol{b}\|$ we get $\|\boldsymbol{b}\| \geq \|\boldsymbol{x}\|/\|\boldsymbol{A}^{-1}\|$ and so

$$\frac{\|\boldsymbol{r}\|}{\|\boldsymbol{b}\|} \leq \|\boldsymbol{A}\| \frac{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|/\|\boldsymbol{A}^{-1}\|} = \boldsymbol{\kappa}(\boldsymbol{A}) \frac{\|\boldsymbol{x} - \tilde{\boldsymbol{x}}\|}{\|\boldsymbol{x}\|}$$

which yields the result after dividing the 2 sides by $\boldsymbol{\kappa}(\boldsymbol{A})$. $\square$

Proof of Theorem 3

Let $D \equiv \|E\| \|y\| + \|e_b\|$ and $\eta \equiv \eta_{E,e_b}(y)$. The theorem states that $\eta = \|r\|/D$. Proof in 2 steps.

First: Any $\Delta A, \Delta b$ pair satisfying (1) is such that $\epsilon \geq \|r\|/D$. Indeed from (1) we have (recall that $r = b - Ay$)

$$Ay + \Delta A y = b + \Delta b \rightarrow r = \Delta A y - \Delta b \rightarrow$$

$$\|r\| \leq \|\Delta A\| \|y\| + \|\Delta b\| \leq \epsilon(\|E\| \|y\| + \|e_b\|) \rightarrow \epsilon \geq \frac{\|r\|}{D}$$

Second: We need to show an instance where the minimum value of $\|r\|/D$ is reached. Take the pair $\Delta A, \Delta b$:

$$\Delta A = \alpha r z^T; \quad \Delta b = \beta r \quad \text{with } \alpha = \frac{\|E\| \|y\|}{D}; \quad \beta = \frac{\|e_b\|}{D}$$

The vector z depends on the norm used - for the 2-norm: $z = y/\|y\|^2$. Here: Proof only for 2-norm

a) We need to verify that first part of (1) is satisfied:

$$
\begin{aligned}
(A + \Delta A)y &= Ay + \alpha r \frac{y^T}{\|y\|^2} y = b - r + \alpha r \\
&= b - (1-\alpha) r = b - \left(1 - \frac{\|E\| \, \|y\|}{\|E\| \, \|y\| + \|e_b\|}\right) r \\
&= b - \frac{\|e_b\|}{D} r = b + \beta r \quad \rightarrow \\
(A + \Delta A)y &= b + \Delta b \quad \leftarrow \text{The desired result}
\end{aligned}
$$

Finally: b) Must now verify that $\|\Delta A\| = \eta\|E\|$ and $\|\Delta b\| = \eta\|e_b\|$. Exercise: Show that $\|uv^T\|_2 = \|u\|_2\|v\|_2$

$$\|\Delta A\| = \frac{|\alpha|}{\|y\|^2}\|ry^T\| = \frac{\|E\|\,\|y\|}{D}\frac{\|r\|\,\|y\|}{\|y\|^2} = \eta\|E\|$$

$$\|\Delta b\| = |\beta|\|r\| = \frac{\|e_b\|}{D}\|r\| = \eta\|e_b\| \qquad QED$$