✍1 Show that each A_k [$A(1:k,1:k)$ in matlab notation] is SPD.

Solution: Let x be any vector in $\mathbb{R}^k$ and consider the vector y of $\mathbb{R}^n$ obtained by stacking x followed by $n-k$ zeros. Then it can be easily seen that : $(A_k x, x) = (Ay, y)$ and since A is SPD then $(Ay, y) > 0$ and therefore $(A_k x, x) > 0$ for any x in $\mathbb{R}^k$. Hence A_k is SPD. □

✍2 Consequence $\det(A_k) > 0$

Solution: This is because the determinant is the product of the eigenvalues which are real positive (see notes). □

✍3 If A is SPD then for any $n \times k$ matrix X of rank k, the matrix $X^T A X$ is SPD.

Solution: For any $v \in \mathbb{R}^k$ we have $(X^T A X v, v) = (AXv, Xv)$. In addition, since X is of full rank, then Xv cannot be zero if v is nonzero. Therefore we have $(AXv, Xv) > 0$. □

✍4 Show that if $A^T = A$ and $(Ax, x) = 0\ \forall x$ then $A = 0$.

Solution: The condition implies that for all x, y we have $(A(x+y), x+y) = 0$. Now expand this as: $(Ax, x) + (Ay, y) + 2(Ax, y) = 0$ for all x, y which shows that $(Ax, y) = 0\ \forall\ x, y$. This implies that $A = 0$ (e.g. take $x = e_j, y = e_i$)... □

✍5 Show: A nonzero matrix A is indefinite iff $\exists\ x, y : (Ax, x)(Ay, y) < 0$.

Solution:

$\boxed{\leftarrow}$ Trivial. The matrix cant be PSD or NSD under the conditon

$\boxed{\rightarrow}$ Need to prove: If A is indefinite then there exist such that $x, y : (Ax, x)(Ay, y) < 0$. Assume contrary is true, i.e., $\forall x, y (Ax, x)(Ay, y) \geq 0$. There is at least one x_0 such that (Ax_0, x_0) is nonzero, otherwise $A = 0$ from previous question. Assume $(Ax_0, x_0) > 0$. Then $\forall y (Ax_0, x_0)(Ay, y) \geq 0$.

which implies $\forall y : (Ay, y) \geq 0$, i.e., A is positive semi-definite. This contradicts the assumption that A is neither positive nor negative semi-defininte $\square$

✍6 The (standard) LU factorization of an SPD matrix A exists.

Solution: This is an immediate consequence of the main theorem on existence (Lec. notes. set #5) and Exercise 1 in this set which showed that $\det(A_k) > 0$ for $k = 1, \cdots, n$. $\square$

Example:

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

✍7 Is A symmetric positive definite?

Solution: Answer is yes because $\det(A_k) > 0$ for $k = 1, 2, 3$. □

✍8 What is the LDL^T factorization of A ?

Solution: The LU factorizatis is:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1/2 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Therefore $A = LDL^T$ where L is as given above and

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \square$$

✍9 What is the Cholesky factorization of A ?

Solution: From the above LDLT factorization we have $A = GG^T$ with

$$G = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix} \qquad \square$$

Gradient of $\psi(x) = (Ax, x)$

In practice exercise # 6 it is asked: Let A be symmetric and $\psi(x) = (Ax, x)$. What is the partial derivative $\frac{\partial \psi(x)}{\partial x_k}$? What is the gradient of ψ?

Solution: First note that

$$\psi(x) = \sum_{i=1}^{n} x_i \left[\sum_{j=1}^{n} a_{ij} x_j \right]$$

and so, using basic rules for derivatives of products:

$$\begin{aligned} \frac{\partial \psi(x)}{\partial x_k} &= \sum_{i=1}^{n} \frac{\partial x_i}{\partial x_k} \left[\sum_{j=1}^{n} a_{ij} x_j \right] + \sum_{i=1}^{n} x_i \left[\frac{\partial x_i}{\partial x_k} \sum_{j=1}^{n} a_{ij} x_j \right] \\ &= \sum_{j=1}^{n} a_{kj} x_j + \sum_{i=1}^{n} x_i a_{ik} \\ &= 2 \sum_{j=1}^{n} a_{kj} x_j \end{aligned}$$

which is nothing but twice the $\boldsymbol{k}$-th component of $\boldsymbol{Ax}$ or $\frac{\partial \psi(x)}{\partial x_k} = \mathbf{2}(\boldsymbol{Ax})_{\boldsymbol{k}}$. Therefore the gradient of $\boldsymbol{\psi}$ is

$$\boldsymbol{\nabla} \boldsymbol{\psi}(\boldsymbol{x}) = \mathbf{2}\boldsymbol{Ax}.$$

A somewhat simpler solution for finding the gradient is to expand $\boldsymbol{\psi}(\boldsymbol{x}+\boldsymbol{\delta}) = (\boldsymbol{A}(\boldsymbol{x}+\boldsymbol{\delta}), (\boldsymbol{x}+\boldsymbol{\delta})) = \ldots$ and to write that the linear term should be of the form $[\boldsymbol{\nabla}\psi]^{\boldsymbol{T}}\, \boldsymbol{\delta}$. $\square$