✍2 Prove that Gram-Schmidt can be completed iff the x_i's are linearly independent.

Solution:

We will show that Gram-Schmidt breaks down iff the x_i's are linearly dependent.

The only way in which GS can break down is if $r_{jj} = \|\hat{q}\|_2$ in line 7 is zero.

The main observation is that the vector $\hat{q}$ at the end of the loop starting in line 4 (ending in line 6) is a linear combination of the x_i's for $i = 1, \cdots, j-1$. [simple proof by induction – omitted.] $\square$

✍3 Cost of Gram-Schmidt?

Solution: Step j of the algorithm costs : $(j-1) \times 2m$ operations for line 3, + $(j-1) \times 2m$ operations for loop in line 4 + $3m$ operations in Lines 7 and 8 together. Total for step j = $c_j = (4j-1)m$. Total over the n columns = $T(n) = (2n^2 + n)m \approx 2n^2 m$.

Note: this is linear in m (number of rows) and quadratic in n (number of columns). □

✍5 What is the cost of solving a linear system with the QR factorization?

Solution: According to the previous question we have a cost of $2n^3$ for the factorization (since $m = n$), to which we need to add the cost of solving a triangular solve $O(n^2)$ and the cost for computing $Q^T b$ which is again $O(n^2)$. In the end the cost is dominated by the QR factorization which is $2n^3$. This is 3 times more expensive than GE. □