✍1 How would you get an orthonormal basis of $\boldsymbol{X}$?

Solution: Just take any basis and orthonormalize it with Gram-Schmidt of Householder QR. $\square$

✍2 Show how you can get a decomposition in which $\boldsymbol{C}$ is lower (or upper) triangular, from the above factorization.

Solution: You first get any factorization in the form shown in Page 9-7 – Then to get an upper triangular $\boldsymbol{C}$ you use the QR factorization $\boldsymbol{C} = \boldsymbol{Q}\boldsymbol{R}$. Then $\boldsymbol{U}$ is replaced by

$$\boldsymbol{U}_{new} = \boldsymbol{U} \times \begin{pmatrix} \boldsymbol{Q} & 0 \\ 0 & \boldsymbol{I} \end{pmatrix}$$

and $\boldsymbol{C}$ is replaced by $\boldsymbol{R}$. To get a lower triangular $\boldsymbol{C}$ you can use the same trick applied to $\boldsymbol{A}^T$ and transpose the final result. $\square$

✍3 How can you get the ULV decomposition by using only the Householder QR factorization (possibly with pivoting)?

Solution: You first get the Householder QR factorization $\boldsymbol{A} = \boldsymbol{Q}_1\boldsymbol{R}_1$ of the matrix $\boldsymbol{A}$. The second step is to perform a Householder QR factorization of the matrix $\boldsymbol{R}_1^T$, so you will get: $\boldsymbol{R}_1^T = \boldsymbol{Q}_2\boldsymbol{R}_2$. The final step is to write:

$$\boldsymbol{A} = \boldsymbol{Q}_1 * \boldsymbol{R}_2^T * \boldsymbol{Q}_2^T \equiv \boldsymbol{U}\boldsymbol{R}\boldsymbol{V}^T$$

where $\boldsymbol{U} = \boldsymbol{Q}_1 \in \mathbb{R}^{m\times m}$; $\boldsymbol{V} = \boldsymbol{Q}_2 \in \mathbb{R}^{n\times n}$; $\boldsymbol{R} = \boldsymbol{R}_2 \in \mathbb{R}^{m\times n}$ $\square$

✍4 In the proof of the SVD decomposition, define $\boldsymbol{U}, \boldsymbol{V}$ as single Householder reflectors.

Solution: We deal with $\boldsymbol{U}$ only [proceed similarly with $\boldsymbol{V}$]. We need a matrix $\boldsymbol{P} = \boldsymbol{I} - 2\boldsymbol{w}\boldsymbol{w}^T$ such that the first column of $\boldsymbol{A}$ is $\boldsymbol{u}_1$ and all columns are orthonormal. The second requirement is satisfied by default since $\boldsymbol{P}$ is unitary. Note that if $\boldsymbol{P}$ is available we will have $\boldsymbol{P}\boldsymbol{u}_1 = \boldsymbol{e}_1$ because $\boldsymbol{P}^2 = \boldsymbol{I}$. Therefore, the wanted $\boldsymbol{w}$ is simply the vector that transforms the vector $\boldsymbol{u}_1$ into $\boldsymbol{\alpha e}_1$. ... $\square$

✍5 How can you obtain the thin SVD from the QR factorization of $\boldsymbol{A}$ and the SVD of an $\boldsymbol{n} \times \boldsymbol{n}$ matrix?

Solution: We first get the thin QR factorization of $\boldsymbol{A}$, namely $\boldsymbol{A} = \boldsymbol{QR}$ where $\boldsymbol{Q} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{R} \in \mathbb{R}^{n \times n}$; Then we can get the SVD $\boldsymbol{R} = \boldsymbol{U_R \Sigma V_R^T}$ of $\boldsymbol{R}$ and this yields:

$$\boldsymbol{A} = \boldsymbol{Q} \times \boldsymbol{U_R \Sigma_R V_R^T} \to \boldsymbol{A} = \boldsymbol{U \Sigma V^T}, \quad \text{with} \quad \boldsymbol{U} = \boldsymbol{Q} \times \boldsymbol{U_R}; \quad \boldsymbol{\Sigma} = \boldsymbol{\Sigma_R}; \quad \boldsymbol{V} = \boldsymbol{V_R}.$$