A few applications of the SVD

Many methods require to approximate the original data (matrix) by a low rank matrix before attempting to solve the original problem

- Regularization methods require the solution of a least-squares linear system $Ax = b$ approximately in the dominant singular space of A

- The Latent Semantic Indexing (LSI) method in information retrieval, performs the "query" in the dominant singular space of A

- Methods utilizing Principal Component Analysis, e.g. Face Recognition.

Commonality: Approximate A (or $A^{\dagger}$) by a lower rank approximation A_k (using dominant singular space) before solving original problem.

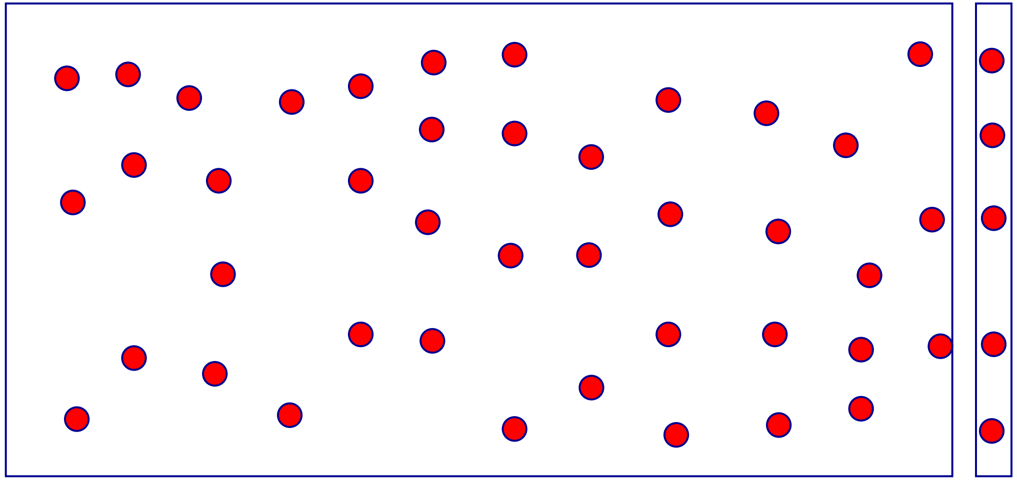
➤ This approximation captures the main features of the data while getting rid of noise and redundancy

Note: Common misconception: 'we need to reduce dimension in order to reduce computational cost'. In reality: using less information often yields better results. This is the problem of overfitting.

➤ Good illustration: Information Retrieval (IR)

Information Retrieval: Vector Space Model

- Given: a collection of documents (columns of a matrix A) and a query vector q.



- Collection represented by an $m \times n$ term by document matrix with $\boxed{a_{ij} = L_{ij} G_i N_j}$
- Queries ('pseudo-documents') q are represented similarly to a column

Vector Space Model - continued

- Problem: find a column of A that best matches q
- Similarity metric: angle between the column and q - Use cosines:

$$\frac{|c^T q|}{\|c\|_2 \|q\|_2}$$

- To rank all documents we need to compute

$$\boxed{s = A^T q}$$

- s = similarity vector.
- Literal matching – not very effective.

Use of the SVD

- Many problems with literal matching: *polysemy*, *synonymy*, ...
- Need to extract intrinsic information – or underlying "semantic" information –
- Solution (LSI): replace matrix $\boldsymbol{A}$ by a low rank approximation using the Singular Value Decomposition (SVD)

$$\boldsymbol{A} = \boldsymbol{U\Sigma V}^T \quad \rightarrow \quad \boldsymbol{A}_k = \boldsymbol{U}_k\boldsymbol{\Sigma}_k\boldsymbol{V}_k^T$$

- $\boldsymbol{U}_k$: term space, $\boldsymbol{V}_k$: document space.
- Refer to this as Truncated SVD (TSVD) approach

New similarity vector:

$$\boldsymbol{s}_k = \boldsymbol{A}_k^T \boldsymbol{q} = \boldsymbol{V}_k \boldsymbol{\Sigma}_k \boldsymbol{U}_k^T \boldsymbol{q}$$

Issues:

- Problem 1: How to select k?
- Problem 2: computational cost (memory + computation)
- Problem 3: updates [e.g. google data changes all the time]
- Not practical for very large sets

LSI : an example

```
%% D1   :   INFANT & TODDLER first aid
%% D2   :   BABIES & CHILDREN's room for your HOME
%% D3   :   CHILD SAFETY at HOME
%% D4   :   Your BABY's HEALTH and SAFETY
%%      :   From INFANT to TODDLER
%% D5   :   BABY PROOFING basics
%% D6   :   Your GUIDE to easy rust PROOFING
%% D7   :   Beanie BABIES collector's GUIDE
%% D8   :   SAFETY GUIDE for CHILD PROOFING your HOME
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% TERMS: 1:BABY  2:CHILD 3:GUIDE 4:HEALTH 5:HOME
%%        6:INFANT 7:PROOFING  8:SAFETY 9:TODDLER
%% Source: Berry and Browne, SIAM., '99
```

- Number of documents: 8
- Number of terms: 9

➤ Raw matrix (before scaling):

$$A = \begin{array}{|cccccccc|l}
d1 & d2 & d3 & d4 & d5 & d6 & d7 & d8 & \\
\hline
 & 1 & & 1 & 1 & & 1 & & bab \\
 & 1 & 1 & & & & & 1 & chi \\
 & & & & & 1 & 1 & 1 & gui \\
 & & & 1 & & & & & hea \\
 & 1 & 1 & & & & & 1 & hom \\
1 & & & 1 & & & & & inf \\
 & & & & 1 & 1 & & 1 & pro \\
 & & 1 & 1 & & & & 1 & saf \\
1 & & & 1 & & & & & tod \\
\hline
\end{array}$$

✍1 Get the anwser to the query `Child Safety`, so

$$q = [0\ 1\ 0\ 0\ 0\ 0\ 0\ 1\ 0]$$

using cosines and then using LSI with $k = 3$.

Dimension reduction

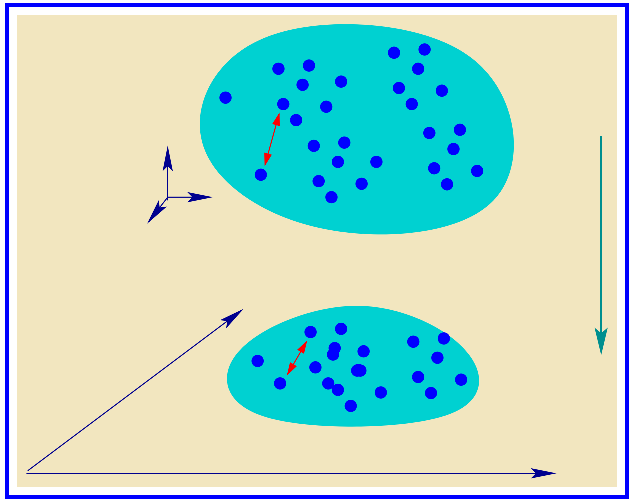
Dimensionality Reduction (DR) techniques pervasive to many applications

➤ Often main goal of dimension reduction is not to reduce computational cost. Instead:

- Dimension reduction used to reduce noise and redundancy in data
- Dimension reduction used to discover patterns (e.g., supervised learning)

➤ Techniques depend on desirable features or application: Preserve angles? Preserve distances? Maximize variance? ..

The problem



- Given $\boldsymbol{d} \ll \boldsymbol{m}$ find a mapping

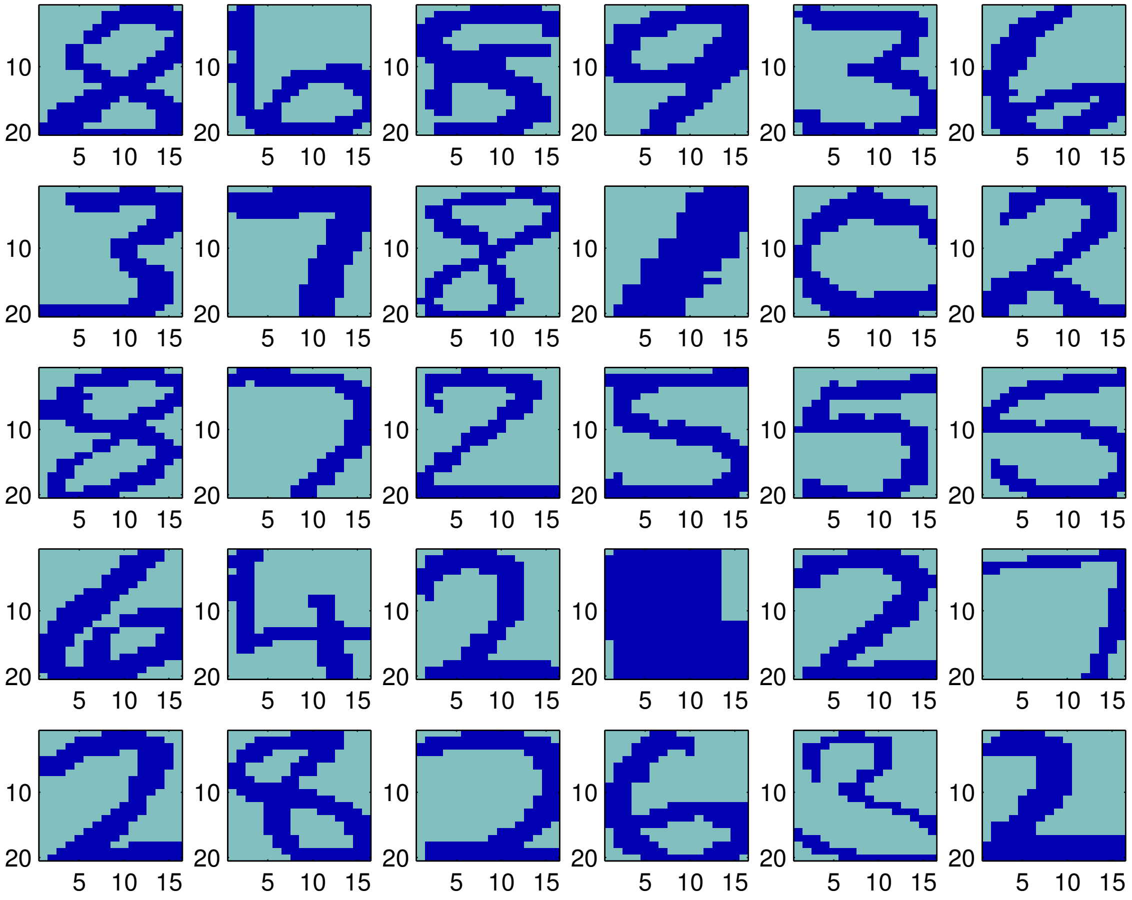
$$\boldsymbol{\Phi} : \boldsymbol{x} \ \in \mathbb{R}^{\boldsymbol{m}} \longrightarrow \boldsymbol{y} \ \in \mathbb{R}^{\boldsymbol{d}}$$

- Mapping may be explicit (e.g., $\boldsymbol{y} = \boldsymbol{V}^T\boldsymbol{x}$)
- Or implicit (nonlinear)

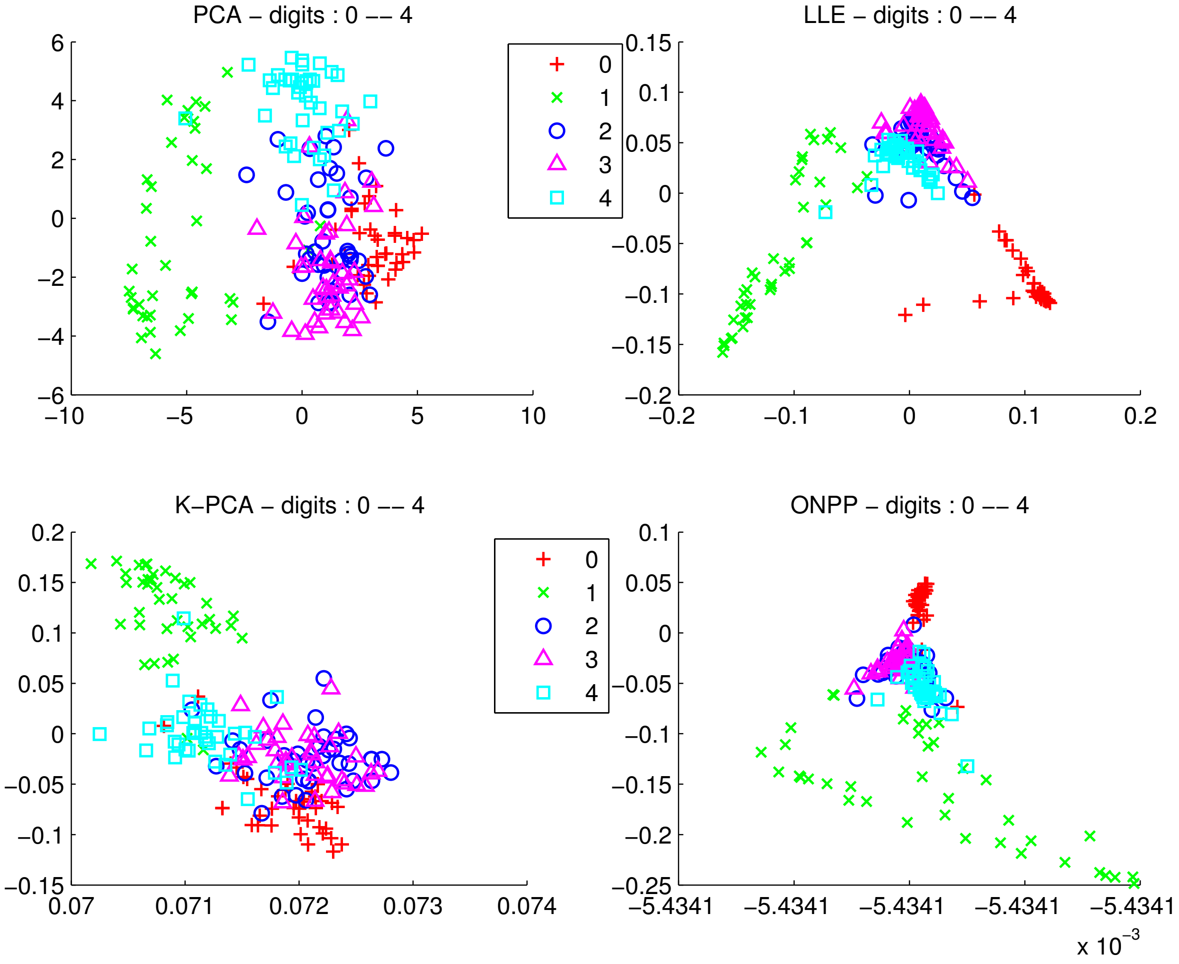
Practically: Find a low-dimensional representation $\boldsymbol{Y} \ \in \ \mathbb{R}^{\boldsymbol{d}\times \boldsymbol{n}}$ of $\boldsymbol{X} \ \in \ \mathbb{R}^{\boldsymbol{m}\times \boldsymbol{n}}$.

- Two classes of methods: (1) projection techniques and (2) nonlinear implicit methods.

Example: Digit images (a sample of 30)



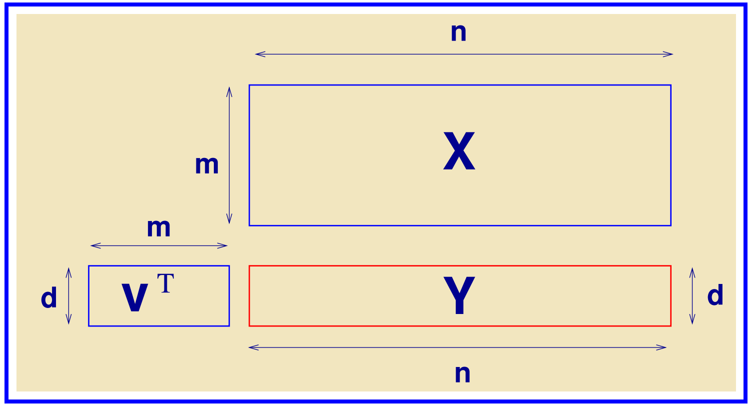
A few 2-D 'reductions':



Projection-based Dimensionality Reduction

Given: a data set $X = [x_1, x_2, \ldots, x_n]$, and d the dimension of the desired reduced space Y.

Want: a linear transformation from X to Y



$$X \in \mathbb{R}^{m\times n}$$
$$V \in \mathbb{R}^{m\times d}$$
$$\boxed{Y = V^\top X}$$
$$\rightarrow \quad Y \in \mathbb{R}^{d\times n}$$

- m-dimens. objects (x_i) 'flattened' to d-dimens. space (y_i)

Problem: Find the best such mapping (optimization) given that the y_i's must satisfy certain constraints

Principal Component Analysis (PCA)

- PCA: find V (orthogonal) so that projected data $Y = V^T X$ has maximum variance

- Maximize over all orthogonal $m \times d$ matrices V:

$$\sum_i \left\| y_i - \frac{1}{n} \sum_j y_j \right\|_2^2 = \cdots = \mathrm{Tr}\left[V^\top \bar{X} \bar{X}^\top V \right]$$

Where: $\bar{X} = [\bar{x}_1, \cdots, \bar{x}_n]$ with $\bar{x}_i = x_i - \mu$, μ = mean.

Solution: $V = \{$ dominant eigenvectors $\}$ of covariance matrix

- i.e., Optimal V = Set of left singular vectors of $\bar{X}$ associated with d largest singular values.

✍2 Show that $\bar{X} = X(I - \frac{1}{n}ee^T)$ (here e = vector of all ones). What does the projector $(I - \frac{1}{n}ee^T)$ do?

✍3 Show that solution V also minimizes 'reconstruction error' ..

$$\sum_i \|\bar{x}_i - VV^T\bar{x}_i\|^2 = \sum_i \|\bar{x}_i - V\bar{y}_i\|^2$$

✍4 .. and that it also maximizes $\sum_{i,j} \|y_i - y_j\|^2$

Matrix Completion Problem

Consider a table of movie ratings. You want to predict missing ratings by assuming commonality (low rank matrix).

	given data			predictions		
movie	Paul	Jane	Ann	Paul	Jane	Ann
Title-1	–1	3	–1	–1.2	1.7	–0.7
Title-2	4	x	3	2.8	–1.2	2.5
Title-3	–3	1	–4	–2.7	1.0	–2.5
Title-4	x	–1	–1	–0.5	–0.3	–0.6
Title-5	3	–2	1	1.8	–1.4	1.4
Title-6	–2	3	x	–1.6	1.8	–1.2
	A			X		

➤ Minimize $\|(X - A)_{\text{mask}}\|_F^2 + \mu\|X\|_*$

"minimize sum-of-squares of deviations from known ratings plus sum of singular values of solution (to reduce the rank)."