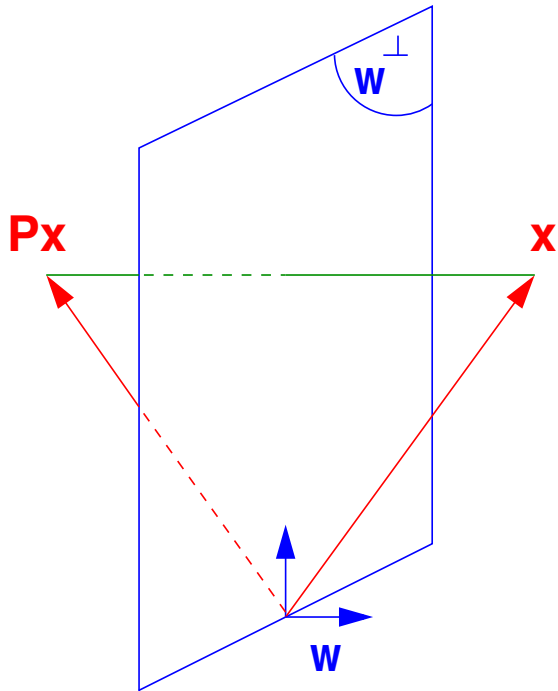


# The Householder QR

➤ Householder reflectors are matrices of the form

$$P = I - 2ww^T,$$

where  $w$  is a unit vector (a vector of 2-norm unity)



Geometrically,  $Px$  represents a mirror image of  $x$  with respect to the hyperplane  $\text{span}\{w\}^\perp$ .

## A few simple properties:

- For real  $w$ :  $P$  is symmetric – It is also **orthogonal** ( $P^T P = I$ ).
  - In the complex case  $P = I - 2ww^H$  is Hermitian and unitary.
  - $P$  can be written as  $P = I - \beta vv^T$  with  $\beta = 2/\|v\|_2^2$ , where  $v$  is a multiple of  $w$ . [storage:  $v$  and  $\beta$ ]
  - $Px$  can be evaluated  $x - \beta(x^T v) \times v$  (op count?)
  - Similarly:  $PA = A - vz^T$  where  $z^T = \beta * v^T * A$
- NOTE: we work in  $\mathbb{R}^m$ , so all vectors are of length  $m$ ,  $P$  is of size  $m \times m$ , etc.
- Next: we will solve a problem that will provide the basic ingredient of the Householder QR factorization.

**Problem 1:** Given a vector  $x \neq 0$ , find  $w$  such that

$$(I - 2ww^T)x = \alpha e_1,$$

where  $\alpha$  is a (free) scalar.

Writing  $(I - \beta vv^T)x = \alpha e_1$  yields  $\beta(v^T x)v = x - \alpha e_1.$

➤ Desired  $w$  is a multiple of  $x - \alpha e_1$ , i.e., we can take :

$$v = x - \alpha e_1$$


➤ To determine  $\alpha$  recall that  $\|(I - 2ww^T)x\|_2 = \|x\|_2$

➤ As a result:  $|\alpha| = \|x\|_2$ , or  $\alpha = \pm \|x\|_2$

➤ Should verify that both signs work, i.e., that in both cases we indeed get  $Px = \alpha e_1$  [exercise]

 1 .. Show that  $(I - \beta vv^T)x = \alpha e_1$  when  $v = x - \alpha e_1$  and  $\alpha = \pm \|x\|_2$ .

 2  $Pw = ?$

 Q: Which sign is best? To reduce cancellation, the resulting  $x - \alpha e_1$  should not be small. So,  $\alpha = -\text{sign}(\xi_1) \|x\|_2$ , where  $\xi_1 = e_1^T x$

$$v = x + \text{sign}(\xi_1) \|x\|_2 e_1 \text{ and } \beta = 2/\|v\|_2^2$$

$$v = \begin{pmatrix} \hat{\xi}_1 \\ \xi_2 \\ \vdots \\ \xi_{m-1} \\ \xi_m \end{pmatrix} \quad \text{with} \quad \hat{\xi}_1 = \begin{cases} \xi_1 + \|x\|_2 & \text{if } \xi_1 > 0 \\ \xi_1 - \|x\|_2 & \text{if } \xi_1 \leq 0 \end{cases}$$

➤ OK, but will yield a negative multiple of  $e_1$  if  $\xi_1 > 0$ .

## Alternative:

- Define  $\sigma = \sum_{i=2}^m \xi_i^2$ .
- Always set  $\hat{\xi}_1 = \xi_1 - \|x\|_2$ . Update OK when  $\xi_1 \leq 0$
- When  $\xi_1 > 0$  compute  $\hat{x}_1$  as

$$\hat{\xi}_1 = \xi_1 - \|x\|_2 = \frac{\xi_1^2 - \|x\|_2^2}{\xi_1 + \|x\|_2} = \frac{-\sigma}{\xi_1 + \|x\|_2}$$

$$\text{So: } \hat{\xi}_1 = \begin{cases} \frac{-\sigma}{\xi_1 + \|x\|_2} & \text{if } \xi_1 > 0 \\ \xi_1 - \|x\|_2 & \text{if } \xi_1 \leq 0 \end{cases}$$

- It is customary to compute a vector  $v$  such that  $v_1 = 1$ . So  $v$  is scaled by its first component.
- If  $\sigma == 0$ , will get  $v = [1; x(2 : m)]$  and  $\beta = 0$ .

## ➤ Matlab function:

```
function [v,bet] = house (x)
%% computes the householder vector for x
m = length(x);
v = [1 ; x(2:m)];
sigma = v(2:m)' * v(2:m);
if (sigma == 0)
    bet = 0;
else
    xnorm = sqrt(x(1)^2 + sigma) ;
    if (x(1) <= 0)
        v(1) = x(1) - xnorm;
    else
        v(1) = -sigma / (x(1) + xnorm) ;
    end
    bet = 2 / (1+sigma/v(1)^2);
    v = v/v(1) ;
end
```

## Problem 2: Generalization.

Want to transform  $x$  into  $y = Px$  where first  $k$  components of  $x$  and  $y$  are the same and  $y_j = 0$  for  $j > k + 1$ . In other words:

**Problem 2:** Given  $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ ,  $x_1 \in \mathbb{R}^k$ ,  $x_2 \in \mathbb{R}^{m-k}$ , find: Householder transform  $P = I - 2ww^T$  such that:  
 $Px = \begin{pmatrix} x_1 \\ \alpha e_1 \end{pmatrix}$  where  $e_1 \in \mathbb{R}^{m-k}$ .

➤ Solution  $w = \begin{pmatrix} 0 \\ \hat{w} \end{pmatrix}$ , where  $\hat{w}$  is s.t.  $(I - 2\hat{w}\hat{w}^T)x_2 = \alpha e_1$

➤ This is because:

$$P = \left[ \begin{array}{c|c} I & 0 \\ \hline 0 & I - 2\hat{w}\hat{w}^T \end{array} \right]$$

## Overall Procedure:

Given an  $m \times n$  matrix  $X$ , find  $w_1, w_2, \dots, w_n$  such that

$$(I - 2w_n w_n^T) \cdots (I - 2w_2 w_2^T) (I - 2w_1 w_1^T) X = R$$

where  $r_{ij} = 0$  for  $i > j$

- First step is easy : select  $w_1$  so that the first column of  $X$  becomes  $\alpha e_1$
- Second step: select  $w_2$  so that  $x_2$  has zeros below 2nd component.
- etc.. After  $k - 1$  steps:  $X_k \equiv P_{k-1} \cdots P_1 X$  has the following shape:



$$X_k = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & \cdots & \cdots & x_{1n} \\ & x_{22} & x_{23} & \cdots & \cdots & \cdots & x_{2n} \\ & & x_{33} & \cdots & \cdots & \cdots & x_{3n} \\ & & & \ddots & \cdots & \cdots & \vdots \\ & & & & x_{kk} & \cdots & \vdots \\ & & & & x_{k+1,k} & \cdots & x_{k+1,n} \\ & & & & \vdots & \vdots & \vdots \\ & & & & x_{m,k} & \cdots & x_{m,n} \end{pmatrix}.$$

- To do: transform this matrix into one which is upper triangular up to the  $k$ -th column...
- ... while leaving the previous columns untouched.

- To leave the first  $k - 1$  columns unchanged  $w$  must have zeros in positions 1 through  $k - 1$ .

$$P_k = I - 2w_k w_k^T, \quad w_k = \frac{v}{\|v\|_2},$$

where the vector  $v$  can be expressed as a Householder vector for a shorter vector using the matlab function `house`,

$$v = \begin{pmatrix} 0 \\ \text{house}(X(k:m, k)) \end{pmatrix}$$

- The result is that work is done on the  $(k : m, k : n)$  submatrix.

## ALGORITHM : 1 ■ Householder QR

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1. For  $k = 1 : n$  do
2.      $[v, \beta] = \text{house}(X(k : m, k))$
3.      $X(k : m, k : n) = (I - \beta v v^T) X(k : m, k : n)$
4.     If  $(k < m)$
5.          $X(k + 1 : m, k) = v(2 : m - k + 1)$
6.     end
7. end

► In the end:

$$X_n = P_n P_{n-1} \dots P_1 X = \text{upper triangular}$$

Yields the factorization:

$$X = QR$$

where:

$$Q = P_1 P_2 \dots P_n \text{ and } R = X_n$$

**Example:**

Apply to system of vectors:

$$X = [x_1, x_2, x_3] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 4 \end{pmatrix}$$

Answer:

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \|x_1\|_2 = 2, v_1 = \begin{pmatrix} 1 + 2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P_1 = I - \frac{2}{v_1^T v_1} v_1 v_1^T = \frac{1}{6} \begin{pmatrix} -3 & -3 & -3 & -3 \\ -3 & 5 & -1 & -1 \\ -3 & -1 & 5 & -1 \\ -3 & -1 & -1 & 5 \end{pmatrix}.$$

$$P_1 X = \begin{pmatrix} -2 & -1 & -2 \\ 0 & 1/3 & -1 \\ 0 & -2/3 & -2 \\ 0 & -2/3 & 3 \end{pmatrix}$$

Next stage:

$$\tilde{x}_2 = \begin{pmatrix} 0 \\ 1/3 \\ -2/3 \\ -2/3 \end{pmatrix}, \|\tilde{x}_2\|_2 = 1, v_2 = \begin{pmatrix} 0 \\ 1/3 + 1 \\ -2/3 \\ -2/3 \end{pmatrix},$$

$$P_2 = I - \frac{2}{v_2^T v_2} v_2 v_2^T = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & 2 & -1 \\ 0 & 2 & -1 & 2 \end{pmatrix},$$

$$P_2 P_1 X = \begin{pmatrix} -2 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 2 \end{pmatrix} \quad \underline{\text{Last stage:}}$$

$$\tilde{x}_3 = \begin{pmatrix} 0 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \quad \|\tilde{x}_3\|_2 = \sqrt{13}, \quad v_3 = \begin{pmatrix} 0 \\ 0 \\ -3 - \sqrt{13} \\ 2 \end{pmatrix},$$

$$P_2 = I - \frac{2}{v_3^T v_3} v_3 v_3^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -.83205 & .55470 \\ 0 & 0 & .55470 & .83205 \end{pmatrix},$$

$$P_3 P_2 P_1 X = \begin{pmatrix} -2 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & \sqrt{13} \\ 0 & 0 & 0 \end{pmatrix} = R,$$

$$P_3 P_2 P_1 = \begin{pmatrix} -.50000 & -.50000 & -.50000 & -.50000 \\ -.50000 & -.50000 & .50000 & .50000 \\ .13868 & -.13868 & -.69338 & .69338 \\ -.69338 & .69338 & -.13868 & .13868 \end{pmatrix}$$

➤ So we end up with the factorization

$$X = \underbrace{P_1 P_2 P_3}_Q R$$

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End Example

MAJOR difference with Gram-Schmidt:  $Q$  is  $m \times m$  and  $R$  is  $m \times n$  (same as  $X$ ). The matrix  $R$  has zeros below the  $n$ -th row. Note also : this factorization always exists.

 3 Cost of Householder QR? Compare with Gram-Schmidt

**Question:**

How to obtain  $X = Q_1 R_1$  where  $Q_1 =$  same size as  $X$  and  $R_1$  is  $n \times n$  (as in MGS)?



**Answer:** simply use the partitioning

$$X = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \rightarrow X = Q_1 R_1$$

- Referred to as the “thin” QR factorization (or “economy-size QR” factorization in matlab)
- How to solve a least-squares problem  $Ax = b$  using the Householder factorization?
- Answer: no need to compute  $Q_1$ . Just apply  $Q^T$  to  $b$ .
- This entails applying the successive Householder reflections to  $b$

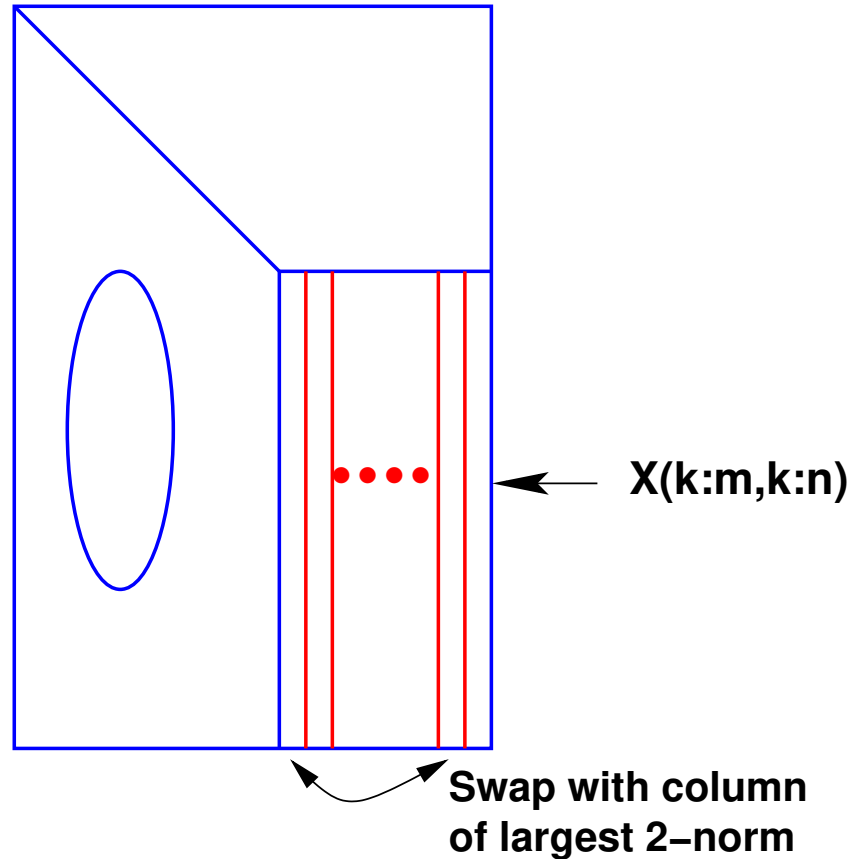
## The rank-deficient case

- Result of Householder QR:  $Q_1$  and  $R_1$  such that  $Q_1 R_1 = X$ . In the rank-deficient case, can have  $\text{span}\{Q_1\} \neq \text{span}\{X\}$  because  $R_1$  may be singular.
- Remedy: Householder QR with column pivoting. Result will be:


$$A\Pi = Q \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix}$$

- $R_{11}$  is nonsingular. So  $\text{rank}(X) = \text{size of } R_{11} = \text{rank}(Q_1)$  and  $Q_1$  and  $X$  span the same subspace.
- $\Pi$  permutes columns of  $X$ .

**Algorithm:** At step  $k$ , active matrix is  $X(k : m, k : n)$ . Swap  $k$ -th column with column of largest 2-norm in  $X(k : m, k : n)$ . If all the columns have zero norm, stop.



**Practical Question:** How to implement this ???

 4 Suppose you know the norms of each column of  $X$  at the start. What happens to each of the norms of  $X(2 : m, j)$  for  $j = 2, \dots, n$ ? Generalize this to step  $k$  and obtain a procedure to inexpensively compute the desired norms at each step.

## *Properties of the QR factorization*

Consider the 'thin' factorization  $A = QR$ , ( $\text{size}(Q) = [m,n] = \text{size}(A)$ ). Assume  $r_{ii} > 0, i = 1, \dots, n$

1. When  $A$  is of full column rank this factorization exists and is unique
2. It satisfies:

$$\text{span}\{a_1, \dots, a_k\} = \text{span}\{q_1, \dots, q_k\}, \quad k = 1, \dots, n$$

3.  $R$  is identical with the Cholesky factor  $G^T$  of  $A^T A$ .

➤ When  $A$  is rank-deficient and Householder with pivoting is used, then

$$\text{Ran}\{Q_1\} = \text{Ran}\{A\}$$


# Givens Rotations and the Givens QR

➤ Givens rotations are matrices of the form:

$$G(i, k, \theta) = \begin{pmatrix} 1 & \dots & 0 & & \dots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & c & \dots & s & \dots & 0 \\ \vdots & \dots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \dots & -s & \dots & c & \dots & 0 \\ \vdots & \dots & \vdots & \dots & \vdots & \dots & \vdots \\ 0 & \dots & 0 & & \dots & & 1 \end{pmatrix} \begin{matrix} \\ \\ i \\ \\ k \\ \\ \end{matrix}$$

with  $c = \cos \theta$  and  $s = \sin \theta$

➤  $G(i, k, \theta)$  represents a rotation in the span of  $e_i$  and  $e_k$

5 Consider the mapping that sends any point  $x$  in  $\mathbb{R}^2$  into a point  $y$  in  $\mathbb{R}^2$  that is **rotated** from  $x$  by an angle  $\theta$ . Find the matrix representing the mapping. [Hint: observe how the canonical basis is transformed.] Show an illustration. What is the mapping corresponding to an angle  $-\theta$ ?

### Main idea of Givens QR

Consider  $y = Gx$  then:

$$\begin{cases} y_i = c * x_i + s * x_k \\ y_k = -s * x_i + c * x_k \\ y_j = x_j \quad \text{for } j \neq i, k \end{cases}$$

➤ Can make  
 $y_k = 0$  by selecting

$$s = x_k/t; \quad c = x_i/t; \quad t = \sqrt{x_i^2 + x_k^2}$$

➤ This is used to introduce zeros in appropriate locations of first column of a matrix  $A$  (for example  $G(m-1, m)$ ,  $G(m-2, m-1)$  etc..  $G(1, 2)$ ). Then similarly for second column, etc.