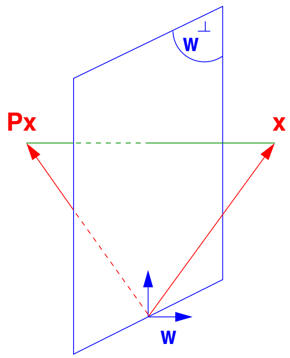
The Householder QR

➤ Householder reflectors are matrices of the form $P = I - 2ww^T,$ where w is a unit vector (a vector of 2-norm unity)



Geometrically, Px represents a mirror image of x with respect to the hyperplane $\mathbf{span}\{w\}^\perp$.

A few simple properties:

- For real w: P is symmetric – It is also orthogonal $(P^TP = I)$.
- In the complex case $P = I - 2ww^H$ is Hermitian and unitary.
- P can be written as $P = I - \beta vv^T$ with $\beta = 2/\|v\|_2^2$, where v is a multiple of w. [storage: v and β]
- Px can be evaluated $x - \beta(x^Tv) \times v$ (op count?)
- Similarly: $PA = A - vz^T$ where $z^T = \beta * v^T * A$

➤ NOTE: we work in $\mathbb{R}^m$, so all vectors are of length m, P is of size $m \times m$, etc.

➤ Next: we will solve a problem that will provide the basic ingredient of the Householder QR factorization.

Problem 1: Given a vector $x \neq 0$, find w such that

$$(I - 2ww^T)x = \alpha e_1,$$

where α is a (free) scalar.

Writing $(I - \beta vv^T)x = \alpha e_1$ yields $\boxed{\beta(v^Tx)\, v = x - \alpha e_1.}$

➤ Desired w is a multiple of $x - \alpha e_1$, i.e., we can take : $v = x - \alpha e_1$

➤ To determine α recall that $\boxed{\|(I - 2ww^T)x\|_2 = \|x\|_2}$

➤ As a result: $|\alpha| = \|x\|_2$, or $\alpha = \pm\|x\|_2$

➤ Should verify that both signs work, i.e., that in both cases we indeed get $Px = \alpha e_1$ [exercise]

✍1 .. Show that $(I - \beta vv^T)x = \alpha e_1$ when $v = x - \alpha e_1$ and $\alpha = \pm\|x\|_2$.

✍2 $Pw = ?$

Q: Which sign is best? To reduce cancellation, the resulting $x - \alpha e_1$ should not be small. So, $\alpha = -\mathrm{sign}(\xi_1)\|x\|_2$, where $\xi_1 = e_1^Tx$

$$v = x + \mathrm{sign}(\xi_1)\|x\|_2 e_1 \text{ and } \beta = 2/\|v\|_2^2$$

$$v = \begin{pmatrix} \hat{\xi}_1 \\ \xi_2 \\ \vdots \\ \xi_{m-1} \\ \xi_m \end{pmatrix} \quad \text{with} \quad \hat{\xi}_1 = \begin{cases} \xi_1 + \|x\|_2 & \text{if } \xi_1 > 0 \\ \xi_1 - \|x\|_2 & \text{if } \xi_1 \le 0 \end{cases}$$

➤ OK, but will yield a negative multiple of e_1 if $\xi_1 > 0$.

Alternative:

- Define $\sigma = \sum_{i=2}^{m} \xi_i^2$.
- Always set $\hat{\xi}_1 = \xi_1 - \|x\|_2$. Update OK when $\xi_1 \leq 0$
- When $\xi_1 > 0$ compute $\hat{x}_1$ as

$$\hat{\xi}_1 = \xi_1 - \|x\|_2 = \frac{\xi_1^2 - \|x\|_2^2}{\xi_1 + \|x\|_2} = \frac{-\sigma}{\xi_1 + \|x\|_2}$$

So: $$\hat{\xi}_1 = \begin{cases} \frac{-\sigma}{\xi_1+\|x\|_2} & \text{if } \xi_1 > 0 \\ \xi_1 - \|x\|_2 & \text{if } \xi_1 \leq 0 \end{cases}$$

- It is customary to compute a vector v such that $v_1 = 1$. So v is scaled by its first component.
- If $\sigma == 0$, wll get $v = [1; x(2:m)]$ and $\beta = 0$.

- Matlab function:

```
function [v,bet] = house (x)
%% computes the householder vector for x
m = length(x);
v = [1 ; x(2:m)];
sigma = v(2:m)' * v(2:m);
if (sigma == 0)
   bet = 0;
else
   xnrm = sqrt(x(1)^2 + sigma) ;
   if (x(1) <= 0)
      v(1) = x(1) - xnrm;
   else
      v(1) = -sigma / (x(1) + xnrm) ;
   end
   bet = 2 / (1+sigma/v(1)^2);
   v = v/v(1) ;
end
```


Problem 2: Generalization.

Want to transform x into $y = Px$ where first k components of x and y are the same and $y_j = 0$ for $j > k+1$. In other words:

Problem 2: Given $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $x_1 \in \mathbb{R}^k, x_2 \in \mathbb{R}^{m-k}$, find: Householder transform $P = I - 2ww^T$ such that:
$Px = \begin{pmatrix} x_1 \\ \alpha e_1 \end{pmatrix}$ where $e_1 \in \mathbb{R}^{m-k}$.

- Solution $w = \begin{pmatrix} 0 \\ \hat{w} \end{pmatrix}$, where $\hat{w}$ is s.t. $(I - 2\hat{w}\hat{w}^T)x_2 = \alpha e_1$
- This is because: $$P = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I - 2\hat{w}\hat{w}^T \end{array}\right]$$

Overall Procedure:

Given an $m \times n$ matrix X, find $w_1, w_2, \ldots, w_n$ such that

$$(I - 2w_n w_n^T) \cdots (I - 2w_2 w_2^T)(I - 2w_1 w_1^T)X = R$$

where $r_{ij} = 0$ for $i > j$

- First step is easy : select w_1 so that the first column of X becomes αe_1
- Second step: select w_2 so that x_2 has zeros below 2nd component.
- etc.. After $k-1$ steps: $X_k \equiv P_{k-1} \ldots P_1 X$ has the following shape:

$$X_k = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & \cdots & \cdots & x_{1n} \\ & x_{22} & x_{23} & \cdots & \cdots & \cdots & x_{2n} \\ & & x_{33} & \cdots & \cdots & \cdots & x_{3n} \\ & & & \ddots & \cdots & \cdots & \vdots \\ & & & & x_{kk} & \cdots & \vdots \\ & & & & x_{k+1,k} & \cdots & x_{k+1,n} \\ & & & & \vdots & \vdots & \vdots \\ & & & & x_{m,k} & \cdots & x_{m,n} \end{pmatrix}.$$

➤ To do: transform this matrix into one which is upper triangular up to the k-th column...

➤ ... while leaving the previous columns untouched.

➤ To leave the first $k-1$ columns unchanged w must have zeros in positions 1 through $k-1$.

$$P_k = I - 2w_k w_k^T, \quad w_k = \frac{v}{\|v\|_2},$$

where the vector v can be expressed as a Householder vector for a shorter vector using the matlab function house,

$$v = \begin{pmatrix} 0 \\ house(X(k:m,k)) \end{pmatrix}$$

➤ The result is that work is done on the $(k:m, k:n)$ submatrix.

ALGORITHM : 1. *Householder QR*

1. *For* $k = 1 : n$ *do*
2. $[v, \beta] = house(X(k:m,k))$
3. $X(k:m, k:n) = (I - \beta v v^T) X(k:m, k:n)$
4. *If* $(k < m)$
5. $X(k+1:m, k) = v(2:m-k+1)$
6. *end*
7. *end*

➤ In the end:

$$X_n = P_n P_{n-1} \dots P_1 X = \text{upper triangular}$$

Yields the factorization: $X = QR$

where: $Q = P_1 P_2 \dots P_n$ and $R = X_n$

Example: Apply to system of vectors:

$$X = [x_1, x_2, x_3] = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 4 \end{pmatrix}$$

Answer:

$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \|x_1\|_2 = 2, v_1 = \begin{pmatrix} 1+2 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v_1 = \begin{pmatrix} 3 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

$$P_1 = I - \frac{2}{v_1^T v_1} v_1 v_1^T = \frac{1}{6}\begin{pmatrix} -3 & -3 & -3 & -3 \\ -3 & 5 & -1 & -1 \\ -3 & -1 & 5 & -1 \\ -3 & -1 & -1 & 5 \end{pmatrix}.$$

$$P_1 X = \begin{pmatrix} -2 & -1 & -2 \\ 0 & 1/3 & -1 \\ 0 & -2/3 & -2 \\ 0 & -2/3 & 3 \end{pmatrix}$$

Next stage:

$$\tilde{x}_2 = \begin{pmatrix} 0 \\ 1/3 \\ -2/3 \\ -2/3 \end{pmatrix}, \ \|\tilde{x}_2\|_2 = 1, \ v_2 = \begin{pmatrix} 0 \\ 1/3+1 \\ -2/3 \\ -2/3 \end{pmatrix},$$

$$P_2 = I - \frac{2}{v_2^T v_2} v_2 v_2^T = \frac{1}{3}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & 2 & -1 \\ 0 & 2 & -1 & 2 \end{pmatrix},$$

$$P_2 P_1 X = \begin{pmatrix} -2 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 2 \end{pmatrix}$$

Last stage:

$$\tilde{x}_3 = \begin{pmatrix} 0 \\ 0 \\ -3 \\ 2 \end{pmatrix}, \ \|\tilde{x}_3\|_2 = \sqrt{13}, \ v_3 = \begin{pmatrix} 0 \\ 0 \\ -3-\sqrt{13} \\ 2 \end{pmatrix},$$

$$P_2 = I - \frac{2}{v_3^T v_3} v_3 v_3^T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -.83205 & .55470 \\ 0 & 0 & .55470 & .83205 \end{pmatrix},$$

$$P_3 P_2 P_1 X = \begin{pmatrix} -2 & -1 & -2 \\ 0 & -1 & 1 \\ 0 & 0 & \sqrt{13} \\ 0 & 0 & 0 \end{pmatrix} = R,$$

$$P_3 P_2 P_1 = \begin{pmatrix} -.50000 & -.50000 & -.50000 & -.50000 \\ -.50000 & -.50000 & .50000 & .50000 \\ .13868 & -.13868 & -.69338 & .69338 \\ -.69338 & .69338 & -.13868 & .13868 \end{pmatrix}$$

➤ So we end up with the factorization $\boxed{X = \underbrace{P_1 P_2 P_3}_{Q} R}$

End Example

MAJOR difference with Gram-Schmidt: Q is $m \times m$ and R is $m \times n$ (same as X). The matrix R has zeros below the n-th row. Note also : this factorization always exists.

✍3 Cost of Householder QR? Compare with Gram-Schmidt

Question: How to obtain $X = Q_1 R_1$ where Q_1 = same size as X and R_1 is $n \times n$ (as in MGS)?

Answer: simply use the partitioning

$$X = \begin{pmatrix} Q_1 & Q_2 \end{pmatrix} \begin{pmatrix} R_1 \\ 0 \end{pmatrix} \quad \rightarrow \quad X = Q_1 R_1$$

- Referred to as the "thin" QR factorization (or "economy-size QR" factorization in matlab)
- How to solve a least-squares problem $Ax = b$ using the Householder factorization?
- Answer: no need to compute Q_1. Just apply Q^T to b.
- This entails applying the successive Householder reflections to b

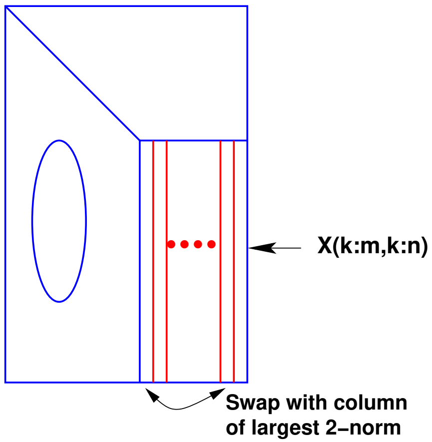
The rank-deficient case

- Result of Householder QR: Q_1 and R_1 such that $Q_1 R_1 = X$. In the rank-deficient case, can have $\mathbf{span}\{Q_1\} \neq \mathbf{span}\{X\}$ because R_1 may be singular.
- Remedy: Householder QR with column pivoting. Result will be:

$$A\Pi = Q \begin{pmatrix} R_{11} & R_{12} \\ 0 & 0 \end{pmatrix}$$

- R_{11} is nonsingular. So rank(X) = size of R_{11} = rank(Q_1) and Q_1 and X span the same subspace.
- Π permutes columns of X.

Algorithm: At step k, active matrix is $X(k:m, k:n)$. Swap k-th column with column of largest 2-norm in $X(k:m, k:n)$. If all the columns have zero norm, stop.



Practical Question: How to implement this ???

✍4 Suppose you know the norms of each column of X at the start. What happens to each of the norms of $X(2:m, j)$ for $j = 2, \cdots, n$? Generalize this to step k and obtain a procedure to inexpensively compute the desired norms at each step.

Properties of the QR factorization

Consider the 'thin' factorization $A = QR$, (size(Q) = [m,n] = size (A)). Assume $r_{ii} > 0, \; i = 1, \ldots, n$

1. When A is of full column rank this factorization exists and is unique
2. It satisfies:

$$\text{span}\{a_1, \cdots, a_k\} = \text{span}\{q_1, \cdots, q_k\}, \quad k = 1, \ldots, n$$

3. R is identical with the Cholesky factor G^T of $A^T A$.

➤ When A in rank-deficient and Householder with pivoting is used, then

$$Ran\{Q_1\} = Ran\{A\}$$

Givens Rotations and the Givens QR

➤ Givens rotations are matrices of the form:

$$G(i,k,\theta) = \begin{array}{c} \begin{pmatrix} 1 & \ldots & 0 & & \ldots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \cdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \cdots & -s & \cdots & c & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & \ldots & 0 & & \ldots & & 1 \end{pmatrix} \end{array} \begin{array}{c} \\ \\ i \\ \\ k \\ \\ \\ \end{array}$$

with $c = \cos\theta$ and $s = \sin\theta$

➤ $G(i,k,\theta)$ represents a rotation in the span of e_i and e_k

✍5 Consider the mapping that sends any point x in $\mathbb{R}^2$ into a point y in $\mathbb{R}^2$ that is rotated from x by an angle θ. Find the matrix representing the mapping. [Hint: observe how the canonical basis is transformed.] Show an illustration. What is the mapping correspoding to an angle $-\theta$?

Main idea of Givens QR

Consider $y = Gx$ then:

$$\begin{cases} y_i &= c * x_i + s * x_k \\ y_k &= -s * x_i + c * x_k \\ y_j &= x_j \quad \text{for} \quad j \neq i, k \end{cases}$$

➤ Can make $y_k = 0$ by selecting

$$s = x_k/t; \; c = x_i/t; \; t = \sqrt{x_i^2 + x_k^2}$$

➤ This is used to introduce zeros in appropriate locations of first column of a matrix A (for example $G(m-1,m)$, $G(m-2,m-1)$ etc..$G(1,2)$). Then similarly for second column, etc.