THE URV & SINGULAR VALUE DECOMPOSITIONS

- **Orthogonal subspaces**
- **Orthogonal projectors, Orthogonal decomposition**
- **The URV decomposition**
- **The Singular Value Decomposition**
- **Properties of the SVD. Relations to eigenvalue problems**

Orthogonal projectors and subspaces

Notation: Given a supspace $\mathcal{X}$ of $\mathbb{R}^m$ define: $\mathcal{X}^\perp = \{y \,|\, y \perp x, \, \forall \, x \in \mathcal{X}\}$

➤ Let $Q = [q_1, \cdots, q_r]$ an orthonormal basis of $\mathcal{X}$

✍1 How would you obtain such a basis?

➤ Then define orthogonal projector $P = QQ^T$

Properties

(a) $P^2 = P$ (b) $(I - P)^2 = I - P$
(c) $Ran(P) = \mathcal{X}$ (d) $Null(P) = \mathcal{X}^\perp$
(e) $Ran(I - P) = Null(P) = \mathcal{X}^\perp$

➤ Note that (b) means that $I - P$ is also a projector

Proof. (a), (b) are trivial

(c): Clearly $Ran(P) = \{x | \, x = QQ^T y, y \in \mathbb{R}^r\} \subseteq \mathcal{X}$. Any $x \in \mathcal{X}$ is of the form $x = Qy, y \in \mathbb{R}^r$. Take $Px = QQ^T(Qy) = Qy = x$. Since $x = Px$, $x \in Ran(P)$. So $\mathcal{X} \subseteq Ran(P)$. In the end $\mathcal{X} = Ran(P)$.

(d): $x \in \mathcal{X}^\perp \leftrightarrow (x, y) = 0, \forall y \in \mathcal{X} \leftrightarrow (x, Qz) = 0, \forall z \in \mathbb{R}^r \leftrightarrow (Q^T x, z) = 0, \forall z \in \mathbb{R}^r \leftrightarrow Q^T x = 0 \leftrightarrow QQ^T x = 0 \leftrightarrow Px = 0.$

(e): Need to show inclusion both ways.

- $x \in Null(P) \leftrightarrow Px = 0 \leftrightarrow (I - P)x = x \rightarrow x \in Ran(I - P)$
- $x \in Ran(I - P) \leftrightarrow \exists y \in \mathbb{R}^m | x = (I - P)y \rightarrow Px = P(I - P)y = 0 \rightarrow x \in Null(P)$ ☐

Result: Any $x \in \mathbb{R}^m$ can be written in a unique way as

$$x = x_1 + x_2, \quad x_1 \in \mathcal{X}, \quad x_2 \in \mathcal{X}^\perp$$

➤ Proof: Just set $x_1 = Px, \quad x_2 = (I - P)x$

➤ Note: $\mathcal{X} \cap \mathcal{X}^\perp = \{0\}$

➤ Therefore: $\mathbb{R}^m = \mathcal{X} \oplus \mathcal{X}^\perp$

➤ Called the *Orthogonal Decomposition*

Orthogonal decomposition

- In other words $\mathbb{R}^m = P\mathbb{R}^m \oplus (I-P)\mathbb{R}^m$ or:
  $\mathbb{R}^m = Ran(P) \oplus Ran(I-P)$ or:
  $\mathbb{R}^m = Ran(P) \oplus Null(P)$ or:
  $\mathbb{R}^m = Ran(P) \oplus Ran(P)^\perp$
- Can complete basis $\{q_1, \cdots, q_r\}$ into orthonormal basis of $\mathbb{R}^m$, $q_{r+1}, \cdots, q_m$
- $\{q_{r+1}, \cdots, q_m\}$ = basis of $\mathcal{X}^\perp$. $\rightarrow$ $dim(\mathcal{X}^\perp) = m - r.$

Four fundamental supspaces - URV decomposition

Let $A \in \mathbb{R}^{m\times n}$ and consider $\mathbf{Ran}(A)^\perp$

Property 1: $\mathbf{Ran}(A)^\perp = Null(A^T)$

Proof: $x \in \mathbf{Ran}(A)^\perp$ iff $(Ay, x) = 0$ for all y; iff $(y, A^Tx) = 0$ for all y ...

Property 2: $\mathbf{Ran}(A^T) = Null(A)^\perp$

- Take $\mathcal{X} = \mathbf{Ran}(A)$ in orthogonal decomoposition. ➤ Result:

$$\mathbb{R}^m = Ran(A) \oplus Null(A^T)$$
$$\mathbb{R}^n = Ran(A^T) \oplus Null(A)$$

4 fundamental subspaces
$Ran(A)$ $Null(A^T)$
$Ran(A^T)$ $Null(A)$

- Express the above with bases for $\mathbb{R}^m$:

$$[\underbrace{u_1, u_2, \cdots, u_r}_{Ran(A)}, \underbrace{u_{r+1}, u_{r+2}, \cdots, u_m}_{Null(A^T)}]$$

and for $\mathbb{R}^n$ $[\underbrace{v_1, v_2, \cdots, v_r}_{Ran(A^T)}, \underbrace{v_{r+1}, v_{r+2}, \cdots, v_n}_{Null(A)}]$

- Observe $u_i^T A v_j = 0$ for $i > r$ or $j > r$. Therefore

$$U^TAV = R = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}_{m\times n} \quad C \in \mathbb{R}^{r\times r} \quad \longrightarrow$$

$$A = URV^T$$

- General class of URV decompositions

- Far from unique.

✍2 Show how you can get a decomposition in which C is lower (or upper) triangular, from the above factorization.

- Can select decomposition so that R is upper triangular $\rightarrow$ URV decomposition.
- Can select decomposition so that R is lower triangular $\rightarrow$ ULV decomposition.
- SVD = special case of URV where R = diagonal

✍3 How can you get the ULV decomposition by using only the Householder QR factorization (possibly with pivoting)? [Hint: you must use Householder twice]

The Singular Value Decomposition (SVD)

Theorem For any matrix $A \in \mathbb{R}^{m\times n}$ there exist unitary matrices $U \in \mathbb{R}^{m\times m}$ and $V \in \mathbb{R}^{n\times n}$ such that

$$A = U\Sigma V^T$$

where Σ is a diagonal matrix with entries $\sigma_{ii} \geq 0$.

$$\sigma_{11} \geq \sigma_{22} \geq \cdots \sigma_{pp} \geq 0 \text{ with } p = \min(n, m)$$

- The σ_{ii}'s are the singular values. Notation change $\sigma_{ii} \longrightarrow \sigma_i$

Proof: Let $\sigma_1 = \|A\|_2 = \max_{x,\|x\|_2=1} \|Ax\|_2$. There exists a pair of unit vectors v_1, u_1 such that

$$Av_1 = \sigma_1 u_1$$

- Complete v_1 into an orthonormal basis of $\mathbb{R}^n$

$$V \equiv [v_1, V_2] = n \times n \text{ unitary}$$

- Complete u_1 into an orthonormal basis of $\mathbb{R}^m$

$$U \equiv [u_1, U_2] = m \times m \text{ unitary}$$

✍4 Define U, V as single Householder reflectors.

- Then, it is easy to show that

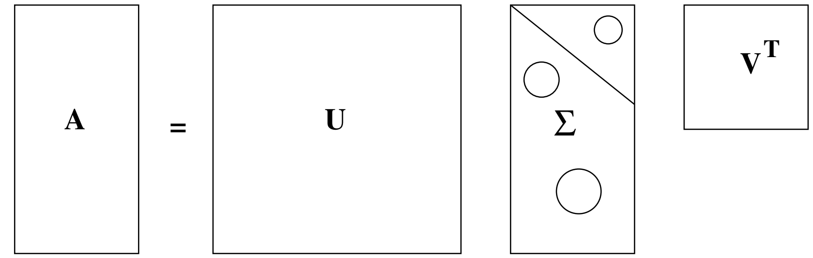
$$AV = U \times \begin{pmatrix} \sigma_1 & w^T \\ 0 & B \end{pmatrix} \rightarrow U^T AV = \begin{pmatrix} \sigma_1 & w^T \\ 0 & B \end{pmatrix} \equiv A_1$$

- Observe that

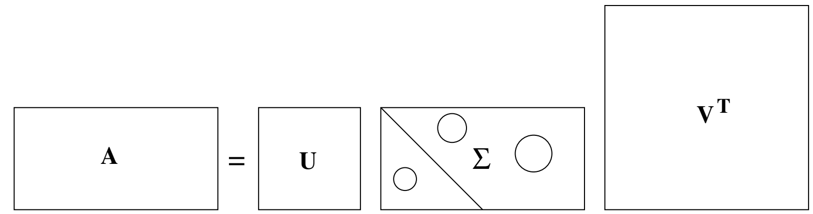
$$\left\| A_1 \begin{pmatrix} \sigma_1 \\ w \end{pmatrix} \right\|_2 \geq \sigma_1^2 + \|w\|^2 = \sqrt{\sigma_1^2 + \|w\|^2} \left\| \begin{pmatrix} \sigma_1 \\ w \end{pmatrix} \right\|_2$$

- This shows that w must be zero [why?]
- Complete the proof by an induction argument. ■

Case 1:



Case 2:



The "thin" SVD

➤ Consider the Case-1. It can be rewritten as

$$A = [U_1 U_2] \begin{pmatrix} \Sigma_1 \\ 0 \end{pmatrix} V^T$$

Which gives:

$$A = U_1 \Sigma_1 V^T$$

where U_1 is $m \times n$ (same shape as A), and Σ_1 and V are $n \times n$

➤ Referred to as the "thin" SVD. Important in practice.

✍5 How can you obtain the thin SVD from the QR factorization of A and the SVD of an $n \times n$ matrix?

A few properties. Assume that

$$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0 \text{ and } \sigma_{r+1} = \cdots = \sigma_p = 0$$

Then:

- $\text{rank}(A) = r$ = number of nonzero singular values.
- $\text{Ran}(A) = \text{span}\{u_1, u_2, \ldots, u_r\}$
- $\text{Null}(A^T) = \text{span}\{u_{r+1}, u_{r+2}, \ldots, u_m\}$
- $\text{Ran}(A^T) = \text{span}\{v_1, v_2, \ldots, v_r\}$
- $\text{Null}(A) = \text{span}\{v_{r+1}, v_{r+2}, \ldots, v_n\}$

Properties of the SVD (continued)

- The matrix A admits the SVD expansion: $$A = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$
- $\|A\|_2 = \sigma_1$ = largest singular value
- $\|A\|_F = \left(\sum_{i=1}^r \sigma_i^2\right)^{1/2}$
- When A is an $n \times n$ nonsingular matrix then $\|A^{-1}\|_2 = 1/\sigma_n$

Theorem [Eckart-Young-Mirsky] Let $k \leq r$ and $A_k = \sum_{i=1}^{k} \sigma_i u_i v_i^T$ then

$$\min_{rank(B)=k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}$$

Proof: First: $\|A - B\|_2 \geq \sigma_{k+1}$, for any rank-$k$ matrix B.

Consider $\mathcal{X} = \text{span}\{v_1, v_2, \cdots, v_{k+1}\}$. Note:

$$dim(Null(B)) = n - k \to Null(B) \cap \mathcal{X} \neq \{0\}$$

[Why?]

Let $x_0 \in Null(B) \cap \mathcal{X}$, $x_0 \neq 0$. Write $x_0 = Vy$. Then

$$\|(A - B)x_0\|_2 = \|Ax_0\|_2 = \|U\Sigma V^T V y\|_2 = \|\Sigma y\|_2$$

But $\|\Sigma y\|_2 \geq \sigma_{k+1} \|x_0\|_2$ (Show this). $\to \|A - B\|_2 \geq \sigma_{k+1}$

Second: take $B = A_k$. Achieves the min. □

Right and Left Singular vectors:

$$Av_i = \sigma_i u_i$$
$$A^T u_j = \sigma_j v_j$$

- v_i's = right singular vectors;
- u_i's = left singular vectors.

- Consequence $A^T A v_i = \sigma_i^2 v_i$ and $A A^T u_i = \sigma_i^2 u_i$
- Right singular vectors (v_i's) are eigenvectors of $A^T A$
- Left singular vectors (u_i's) are eigenvectors of $A A^T$
- Possible to get the SVD from eigenvectors of AA^T and $A^T A$ – but: difficulties due to non-uniqueness of the SVD

Define the $r \times r$ matrix

$$\Sigma_1 = \text{diag}(\sigma_1, \ldots, \sigma_r)$$

- Let $A \in \mathbb{R}^{m\times n}$ and consider $A^T A$ ($\in \mathbb{R}^{n\times n}$):

$$A^T A = V\Sigma^T \Sigma V^T \rightarrow A^T A = V \underbrace{\begin{pmatrix} \Sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix}}_{n\times n} V^T$$

- This gives the spectral decomposition of $A^T A$.

- Similarly, U gives the eigenvectors of AA^T.

$$AA^T = U \underbrace{\begin{pmatrix} \Sigma_1^2 & 0 \\ 0 & 0 \end{pmatrix}}_{m\times m} U^T$$

Important:

$A^T A = V D_1 V^T$ and $AA^T = U D_2 U^T$ give the SVD factors U, V up to signs!