# Symmetric Eigenvalue Problems

- The symmetric eigenvalue problem: basic facts
- Min-Max theorem -
- Inertia of matrices
- Bisection algorithm

# *The symmetric eigenvalue problem: Basic facts*

➤ Consider the Schur form of a real symmetric matrix $A$:

$$A = QRQ^H$$

Since $A^H = A$ then $R = R^H$ ➤

Eigenvalues of $A$ are real

and

There is an orthonormal basis of eigenvectors of $A$

In addition, $Q$ can be taken to be real when $A$ is real.

$$(A - \lambda I)(u + iv) = 0 \to (A - \lambda I)u = 0 \; \& \; (A - \lambda I)v = 0$$

➤ Can select eigenvector to be either $u$ or $v$, whichever is $\neq 0$.

# *The min-max theorem (Courant-Fischer)*

Label eigenvalues decreasingly: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

The eigenvalues of a Hermitian matrix $A$ are characterized by the relation

$$\lambda_k = \max_{S,\ \dim(S)=k} \quad \min_{x \in S, x \neq 0} \frac{(Ax, x)}{(x, x)}$$

*Proof:* Preparation: Since $A$ is symmetric real (or Hermitian complex) there is an orthonormal basis of eigenvectors $u_1, u_2, \cdots, u_n$. Express any vector $x$ in this basis as $x = \sum_{i=1}^{n} \alpha_i u_i$. Then : $(Ax, x)/(x, x) = [\sum \lambda_i |\alpha_i|^2]/[\sum |\alpha_i|^2]$.

(a) Let $S$ be any subspace of dimension $k$ and let $\mathcal{W} = \text{span}\{u_k, u_{k+1}, \cdots, u_n\}$. A dimension argument (used before) shows that $S \cap \mathcal{W} \neq \{0\}$. So there is a non-zero $x_w$ in $S \cap \mathcal{W}$.

➤ Express this $x_w$ in the eigenbasis as $x_w = \sum_{i=k}^{n} \alpha_i u_i$. Then since $\lambda_i \leq \lambda_k$ for $i \geq k$ we have:

$$\frac{(Ax_w, x_w)}{(x_w, x_w)} = \frac{\sum_{i=k}^{n} \lambda_i |\alpha_i|^2}{\sum_{i=k}^{n} |\alpha_i|^2} \leq \lambda_k$$

Thus, for any subspace $S$ of dim. $k$ we have $\min_{x \in S, x \neq 0} (Ax, x)/(x, x) \leq \lambda_k$.

(b) We now take $S_* = \mathbf{span}\{u_1, u_2, \cdots, u_k\}$. Since $\lambda_i \geq \lambda_k$ for $i \leq k$, for this particular subspace we have:

$$\min_{x \,\in\, S_*,\, x \neq 0} \frac{(Ax, x)}{(x, x)} = \min_{x \,\in\, S_*,\, x \neq 0} \frac{\sum_{i=1}^{k} \lambda_i |\alpha_i|^2}{\sum_{i=k}^{n} |\alpha_i|^2} = \lambda_k.$$

(c) The results of (a) and (b) imply that the max over all subspaces $S$ of dim. $k$ of $\min_{x \in S, x \neq 0} (Ax, x)/(x, x)$ is equal to $\lambda_k$ $\square$

- Consequences:

$$\lambda_1 = \max_{x \neq 0} \frac{(Ax, x)}{(x, x)} \qquad \lambda_n = \min_{x \neq 0} \frac{(Ax, x)}{(x, x)}$$

- Actually 4 versions of the same theorem. 2nd version:

$$\lambda_k = \min_{S,\ \dim(S)=n-k+1} \quad \max_{x \in S, x \neq 0} \frac{(Ax, x)}{(x, x)}$$

- Other 2 versions come from ordering eigenvalues increasingly instead of decreasingly.

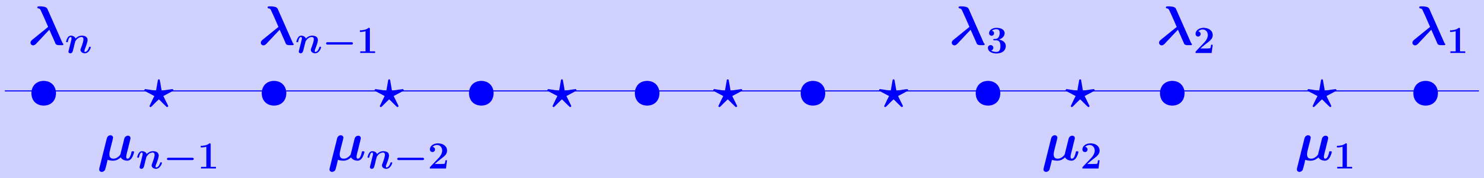
✍1 Write down all 4 versions of the theorem

✍2 Use the min-max theorem to show that $\|A\|_2 = \sigma_1(A)$ - the largest singular value of $A$.

➤ Interlacing Theorem: Denote the $k \times k$ principal submatrix of $A$ as $A_k$, with eigenvalues $\{\lambda_i^{[k]}\}_{i=1}^k$. Then

$$\lambda_1^{[k]} \geq \lambda_1^{[k-1]} \geq \lambda_2^{[k]} \geq \lambda_2^{[k-1]} \geq \cdots \lambda_{k-1}^{[k-1]} \geq \lambda_k^{[k]}$$

***Example:*** $\lambda_i$'s = eigenvalues of $A$, $\mu_i$'s = eigenvalues of $A_{n-1}$:




➤ Many uses.

➤ For example: interlacing theorem for roots of orthogonal polynomials

# The Law of inertia (real symmetric matrices)

➤ Inertia of a matrix = [m, z, p] with $m$ = number of $< 0$ eigenvalues, $z$ = number of zero eigenvalues, and $p$ = number of $> 0$ eigenvalues.

**Sylvester's Law of inertia:**

*If $X \in \mathbb{R}^{n\times n}$ is nonsingular, then $A$ and $X^T A X$ have the same inertia.*

➤ Terminology: $X^T A X$ is congruent to $A$

✍3 Suppose that $A = LDL^T$ where $L$ is unit lower triangular, and $D$ diagonal. How many negative eigenvalues does $A$ have?

✍4 Assume that $A$ is tridiagonal. How many operations are required to determine the number of negative eigenvalues of $A$?

✍5 Devise an algorithm based on the inertia theorem to compute the $i$-th eigenvalue of a tridiagonal matrix.

✍6 Let $F \in \mathbb{R}^{m \times n}$, with $n < m$, and $F$ of rank $n$.

What is the inertia of the matrix on the right:
[Hint: use a block LU factorization]

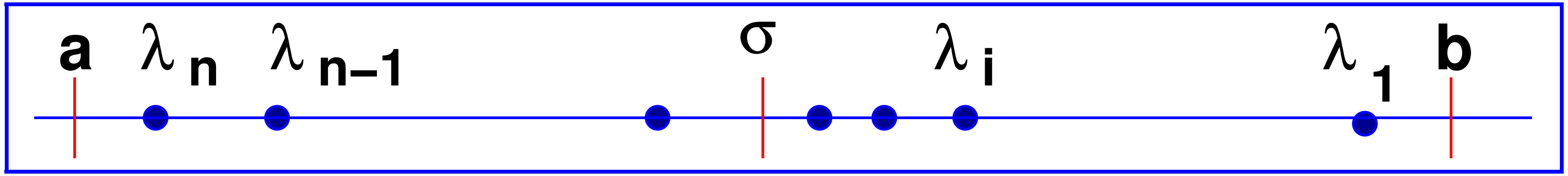
$$\begin{pmatrix} I & F \\ F^T & 0 \end{pmatrix}$$

➤ Note 1: Converse result also true: If $A$ and $B$ have same inertia they are congruent. [This part is easy to show]

➤ Note 2: result also true for (complex) Hermitian matrices ($X^H A X$ has same inertia as $A$).

## Bisection algorithm for tridiagonal matrices:

- Goal: to compute $i$-th eigenvalue of $A$ (tridiagonal)
- Get interval $[a, b]$ containing spectrum [Gerschgorin]: $a \le \lambda_n \le \cdots \le \lambda_1 \le b$
- Let $\sigma = (a + b)/2$ = middle of interval
- Calculate $p =$ number of positive eigenvalues of $A - \sigma I$
  - If $p \ge i$ then $\lambda_i \in (\sigma, b] \rightarrow$ set $a := \sigma$




  - Else then $\lambda_i \in [a, \sigma] \rightarrow$ set $b := \sigma$
- Repeat until $b - a$ is small enough.