Symmetric Eigenvalue Problems

- **The symmetric eigenvalue problem: basic facts**
- **Min-Max theorem -**
- **Inertia of matrices**
- **Bisection algorithm**

The symmetric eigenvalue problem: Basic facts

➤ Consider the Schur form of a real symmetric matrix A:

$$A = QRQ^H$$

Since $A^H = A$ then $R = R^H$ ➤

Eigenvalues of A are real

and

There is an orthonormal basis of eigenvectors of A

In addition, Q can be taken to be real when A is real.

$$(A - \lambda I)(u + iv) = 0 \rightarrow (A - \lambda I)u = 0 \ \& \ (A - \lambda I)v = 0$$

➤ Can select eigenvector to be either u or v, whichever is $\neq 0$.

The min-max theorem (Courant-Fischer)

Label eigenvalues decreasingly: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$

The eigenvalues of a Hermitian matrix A are characterized by the relation

$$\lambda_k = \max_{S,\ \dim(S)=k} \ \min_{x \in S, x \neq 0} \frac{(Ax, x)}{(x, x)}$$

Proof: Preparation: Since A is symmetric real (or Hermitian complex) there is an orthonormal basis of eigenvectors $u_1, u_2, \cdots, u_n$. Express any vector x in this basis as $x = \sum_{i=1}^n \alpha_i u_i$. Then : $(Ax, x)/(x, x) = [\sum \lambda_i |\alpha_i|^2]/[\sum |\alpha_i|^2]$.

(a) Let S be any subspace of dimension k and let $\mathcal{W} = \mathbf{span}\{u_k, u_{k+1}, \cdots, u_n\}$. A dimension argument (used before) shows that $S \cap \mathcal{W} \neq \{0\}$. So there is a non-zero x_w in $S \cap \mathcal{W}$.

➤ Express this x_w in the eigenbasis as $x_w = \sum_{i=k}^n \alpha_i u_i$. Then since $\lambda_i \leq \lambda_k$ for $i \geq k$ we have:

$$\frac{(Ax_w, x_w)}{(x_w, x_w)} = \frac{\sum_{i=k}^n \lambda_i |\alpha_i|^2}{\sum_{i=k}^n |\alpha_i|^2} \leq \lambda_k$$

Thus, for any subspace S of dim. k we have $\min_{x \in S, x \neq 0} (Ax, x)/(x, x) \leq \lambda_k$.

(b) We now take $S_* = \mathbf{span}\{u_1, u_2, \cdots, u_k\}$. Since $\lambda_i \geq \lambda_k$ for $i \leq k$, for this particular subspace we have:

$$\min_{x \,\in\, S_*,\, x \neq 0} \frac{(Ax, x)}{(x, x)} = \min_{x \,\in\, S_*,\, x \neq 0} \frac{\sum_{i=1}^k \lambda_i |\alpha_i|^2}{\sum_{i=k}^n |\alpha_i|^2} = \lambda_k.$$

(c) The results of (a) and (b) imply that the max over all subspaces S of dim. k of $\min_{x \in S, x \neq 0} (Ax, x)/(x, x)$ is equal to λ_k ☐

➤ Consequences:

$$\lambda_1 = \max_{x \neq 0} \frac{(Ax, x)}{(x, x)} \qquad \lambda_n = \min_{x \neq 0} \frac{(Ax, x)}{(x, x)}$$

➤ Actually 4 versions of the same theorem. 2nd version:

$$\lambda_k = \min_{S,\ \dim(S)=n-k+1} \quad \max_{x \in S, x \neq 0} \frac{(Ax, x)}{(x, x)}$$

➤ Other 2 versions come from ordering eigenvalues increasingly instead of decreasingly.

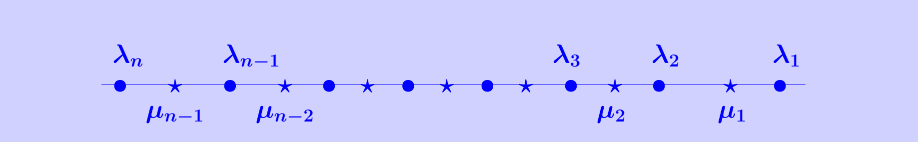
✍1 Write down all 4 versions of the theorem

✍2 Use the min-max theorem to show that $\|A\|_2 = \sigma_1(A)$ - the largest singular value of A.

➤ Interlacing Theorem: Denote the $k \times k$ principal submatrix of A as A_k, with eigenvalues $\{\lambda_i^{[k]}\}_{i=1}^k$. Then

$$\lambda_1^{[k]} \geq \lambda_1^{[k-1]} \geq \lambda_2^{[k]} \geq \lambda_2^{[k-1]} \geq \cdots \lambda_{k-1}^{[k-1]} \geq \lambda_k^{[k]}$$

Example: λ_i's = eigenvalues of A, μ_i's = eigenvalues of A_{n-1}:



➤ Many uses.

➤ For example: interlacing theorem for roots of orthogonal polynomials

The Law of inertia (real symmetric matrices)

➤ Inertia of a matrix = [m, z, p] with $m =$ number of < 0 eigenvalues, $z =$ number of zero eigenvalues, and $p =$ number of > 0 eigenvalues.

Sylvester's Law of inertia: *If $X \in \mathbb{R}^{n \times n}$ is nonsingular, then A and $X^T AX$ have the same inertia.*

➤ Terminology: $X^T AX$ is congruent to A

✍3 Suppose that $A = LDL^T$ where L is unit lower triangular, and D diagonal. How many negative eigenvalues does A have?

✍4 Assume that A is tridiagonal. How many operations are required to determine the number of negative eigenvalues of A?

✍5 Devise an algorithm based on the inertia theorem to compute the i-th eigenvalue of a tridiagonal matrix.

✍6 Let $F \in \mathbb{R}^{m \times n}$, with $n < m$, and F of rank n.
What is the inertia of the matrix on the right:
[Hint: use a block LU factorization]

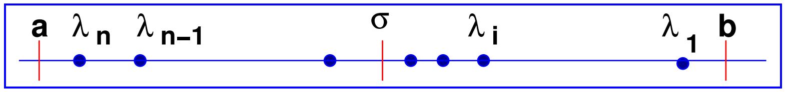
$$\begin{pmatrix} I & F \\ F^T & 0 \end{pmatrix}$$

➤ Note 1: Converse result also true: If A and B have same inertia they are congruent. [This part is easy to show]

➤ Note 2: result also true for (complex) Hermitian matrices ($X^H AX$ has same inertia as A).

Bisection algorithm for tridiagonal matrices:

- Goal: to compute i-th eigenvalue of A (tridiagonal)
- Get interval $[a, b]$ containing spectrum [Gerschgorin]: $a \le \lambda_n \le \cdots \le \lambda_1 \le b$
- Let $\sigma = (a + b)/2$ = middle of interval
- Calculate $p =$ number of positive eigenvalues of $A - \sigma I$
  - If $p \ge i$ then $\lambda_i \in \ (\sigma,\ b] \rightarrow$ set $a := \sigma$



  - Else then $\lambda_i \in \ [a,\ \sigma] \rightarrow$ set $b := \sigma$
- Repeat until $b - a$ is small enough.