✍1 In the proof of the SVD decomposition, define U, V as single Householder reflectors.

Solution: We deal with U only [proceed similarly with V]. We need a matrix $P = I - 2ww^T$ such that the first column of A is u_1 and all columns are orthonormal. The second requirement is satisfied by default since P is unitary. Note that if P is available we will have $Pu_1 = e_1$ because $P^2 = I$. Therefore, the wanted w is simply the vector that transforms the vector u_1 into αe_1. ... $\square$

✍2 How can you obtain the thin SVD from the QR factorization of A and the SVD of an $n \times n$ matrix?

Solution: We first get the thin QR factorization of A, namely $A = QR$ where $Q \in \mathbb{R}^{m\times n}$ and $R \in \mathbb{R}^{n\times n}$; Then we can get the SVD $R = U_R \Sigma V_R^T$ of R and this yields:

$$A = Q \times U_R \Sigma_R V_R^T \rightarrow A = U \Sigma V^T, \quad \text{with} \quad U = Q \times U_R; \quad \Sigma = \Sigma_R;$$