


 2 If $\mathbf{A} \in \mathbb{R}^{m \times n}$ what are the dimensions of \mathbf{A}^\dagger ?, $\mathbf{A}^\dagger \mathbf{A}$?, $\mathbf{A} \mathbf{A}^\dagger$?

Solution: The dimension of $\mathbf{A}^\dagger \mathbf{A}$ is $n \times m$ and so $\mathbf{A}^\dagger \mathbf{A}$ is of size $n \times n$. Similarly, $\mathbf{A} \mathbf{A}^\dagger$ is of size $m \times m$. \square

 3 Show that $\mathbf{A}^\dagger \mathbf{A}$ is an orthogonal projector. What are its range and null-space?

Solution: One way to do this is to use the rank-one expansion: $\mathbf{A} = \sum \sigma_i \mathbf{u}_i \mathbf{v}_i^T$. Then $\mathbf{A}^\dagger = \sum \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^T$ and therefore,

$$\mathbf{A}^\dagger \mathbf{A} = \left[\sum_{i=1}^r \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^T \right] \times \left[\sum_{j=1}^r \sigma_j \mathbf{u}_j \mathbf{v}_j^T \right] = \sum_{j=1}^r \mathbf{v}_j \mathbf{v}_j^T$$

which is a projector. \square

 4 Same question for $\mathbf{A} \mathbf{A}^\dagger$..

Solution: In this case we have

$$\mathbf{A} \mathbf{A}^\dagger = \left[\sum_{j=1}^r \sigma_j \mathbf{u}_j \mathbf{v}_j^T \right] \left[\sum_{i=1}^r \frac{1}{\sigma_i} \mathbf{v}_i \mathbf{u}_i^T \right] \times = \sum_{j=1}^r \mathbf{u}_j \mathbf{u}_j^T$$

which is an orthogonal projector. \square



Consider the matrix:

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix}$$

- Compute the singular value decomposition of A

Solution: The nonzero singular values of A are the square roots of the eigenvalues of

$$AA^T = \begin{pmatrix} 5 & -4 \\ -4 & 5 \end{pmatrix}$$

These eigenvalues are 5 ± 4 and so $\sigma_1 = 3, \sigma_2 = 1$.

The matrix U of the left singular vectors is the matrix

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

If $A = U\Sigma V^T$, then $U' * A = \Sigma V^T$. Therefore to get V we use the relation: $V^T = \Sigma^{-1} * U' * A$. We have

$$U' * A = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 4 & -1 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow V^T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1/3 & 0 & 4/3 & -1/3 \\ 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow$$



- Find the matrix B of rank 1 which is the closest to A in 2-norm

sense.

Solution: This is obtained by setting σ_2 to zero in the SVD - or - equivalently as $B = \sigma_1 u_1 v_1^T$. You will find

$$B = \begin{pmatrix} 1/2 & 0 & 2 & -1/2 \\ -1/2 & 0 & -2 & 1/2 \end{pmatrix}$$



 6 Show that r_ϵ equals the number sing. values that are $> \epsilon$

Solution: This result is based on the following easy-to-prove extension of the Young=Eckhart theorem:

$$\min_{rank(B) \leq k} \|A - B\|_2 = \|A - A_k\|_2 = \sigma_{k+1}$$

which implies that if $\|A - B\|_2 < \sigma_{k+1}$ then $rank(B)$ must be $> k$ - or equivalently:

$$\|A - B\|_2 < \sigma_k \rightarrow rank(B) \geq k.$$

Let k be the number that satisfies $\sigma_{k+1} \leq \epsilon < \sigma_k$ - which is the number of sing. values that are $> \epsilon$. Then we see from the above that $\|A - B\|_2 \leq \epsilon$ implies that $rank(B) \geq k$. The smallest possible rank for B is precisely the integer k defined above. 