

 1 Consider

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}$$

Eigenvalues of A ? their algebraic multiplicities? their geometric multiplicities? Is one a semi-simple eigenvalue?

Solution: The eigenvalues of A are 1, and 2. The algebraic multiplicity of 1 is 2. To get the geometric multiplicity of the eigenvalue $\lambda = 1$ we need to eigenvectors. For this we need to solve:

$$\begin{pmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} u = 0.$$

There is only one solution vector (up to a product by a scalar) namely:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

So the geometric multiplicity is one. ☐

 2 Same questions if a_{33} is replaced by one.

Solution: The matrix become

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and now we have one eigenvalue algebraic multiplicity 3.

To get the geometric multiplicity of the eigenvalue $\lambda = 1$ we need to eigenvectors. For this we need to solve:

$$\begin{pmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} u = 0.$$

we still get a geometric mult. of 1. $\square$

 3 Same questions if in addition a_{12} is replaced by zero.

Solution: Solution: The matrix become

$$A = \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

and we also have one eigenvalue with algebraic multiplicity 3. The

geometric multiplicity increases to 2. $\square$

 4 Show that there is at least one eigenvalue and eigenvector of A :
 $Ax = \lambda x$, with $\|x\|_2 = 1$

Solution: This comes from the fact that the equation $P_A(\lambda) = \det(A - \lambda I) = 0$ is a polynomial equation and as such it must have at least one root - a well-known result. $\square$

 5 There is a unitary transformation P such that $Px = e_1$. How do you define P ?

Solution: This is just the Householder transform.. See Lecture notes set number 8. $\square$

 6 Show that $PAP^H = \left(\begin{array}{c|c} \lambda & ** \\ \hline 0 & A_2 \end{array} \right)$.

Solution: This is equivalent to showing that $PAP^H e_1 = \lambda e_1$. We have

$$PAP^H e_1 = PA P e_1 = P(Ax) = P(\lambda x) = \lambda Px = \lambda e_1$$

$\square$

 9 Another proof altogether: use Jordan form of A and QR factor-

ization **Solution:** Jordan form:

$$A = X J X^{-1}$$

Let $X = Q R_0$ then:

$$A = Q R_0 J R_0^{-1} Q^H \equiv Q R Q^H \quad \text{with} \quad R = R_0 J R_0^{-1}$$

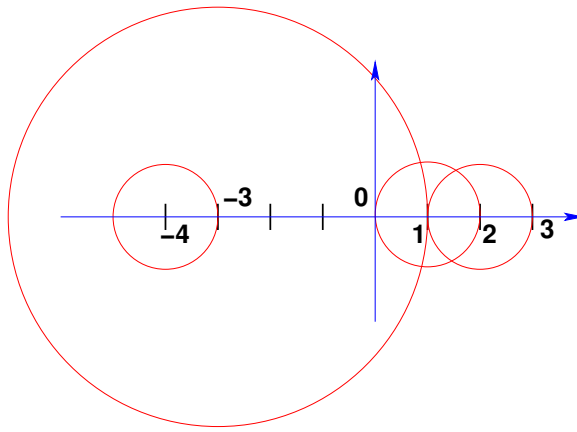


 10 Find a region of the complex plane where the eigenvalues of the following matrix are located:

$$A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & -2 & -3 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & -4 \end{pmatrix}$$

Solution: Use Gershgorin's theorem. There are 4 disks:

$$\begin{aligned} D_1 &= D(1, 1); & D_2 &= D(2, 1) \\ D_3 &= D(-3, 4); & D_4 &= D(-4, 1) \end{aligned}$$



The last disk is included in the 3rd. The spectrum is included in the union of the 3 other disks. $\square$

Additional notes.

In discussing Gerschgorin theorem it was stated:

➤ Refinement: if disks are all disjoint then each of them contains one eigenvalue

Question: Why?

Solution:

Consider the matrix $A(t) = D + t(A - D)$ where D is the diagonal of A . Note $A(0) = D$, $A(1) = A$. Consider the n disks as t varies from $t = 0$ to $t = 1$. When $t = 0$ each disk contains exactly one eigenvalue. As t increases (in a continuous way) from 0 to one – each disk will still contain one eigenvalue - by a continuity argument [you

cannot have an eigenvalue jump suddenly - from one disk to another - this would be a discontinuous behavior]. The argument can be adapted to the case where two disks touch each other at one point (only): it is now possible to have two eigenvalues at the intersection of the disks - coming from each of the two disks.