✍2 Use the min-max theorem to show that $\|A\|_2 = \sigma_1(A)$ - the largest singular value of A.

Solution: This comes from the fact that:

$$\begin{aligned}\|A\|_2^2 &= \max_{x\neq 0} \frac{\|Ax\|_2^2}{\|x\|_2^2} \\ &= \max_{x\neq 0} \frac{(Ax, Ax)}{(x, x)} \\ &= \max_{x\neq 0} \frac{(A^T Ax, A)}{(x, x)} \\ &= \lambda_{max}(A^T A) \\ &= \sigma_1^2\end{aligned}$$

$\square$

✍3 Suppose that $A = LDL^T$ where L is unit lower triangular, and D diagonal. How many negative eigenvalues does A have?

Solution: It has as many negative eigenvalues as there are negative entries in D $\square$

✍4 Assume that A is tridiagonal. How many operations are required to determine the number of negative eigenvalues of A?

Solution: The rough answer is $O(n)$ – because an LU (and therefore LDLT) factorization costs $O(n)$. Based on doing the LU factorization of a triagonal matrix, a more accurate answer is $3n$ operations. $\square$

✍5 Devise an algorithm based on the inertia theorem to compute the i-th eigenvalue of a tridiagonal matrix.

Solution: Here is a matlab script:

```
   function [sigma] = bisect(d, b, i, tol)
%% function [sigma] = bisect(d, b, i, tol)
%% d   = diagonal of T
%% b   = co-diagonal
%% i   = compute i-th eigenvalue
%% tol = tolerance used for stopping
     b(1) = 0;
   n = length(d);
 %%-------------------- guershgorin
   tmin = d(n) - abs(b(n));
   tmax = d(n) + abs(b(n));
   for j=1:n-1
     rho = abs(b(j)) + abs(b(j+1));
     tmin = min(tmin, d(j)-rho);
     tmax = max(tmax, d(j)+rho);
   end
   tol = tol*(tmax-tmin);
   for iter=1:100
     sigma = 0.5*(tmin+tmax);
     count = sturm(d, b, sigma);
     if (count >= i)
       tmin = sigma;
     else
       tmax = sigma;
     end
     if (tmax - tmin) < tol
       break
     end
   end
```

☐

✍6 What is the inertia of the matrix

$$\begin{pmatrix} I & F \\ F^T & 0 \end{pmatrix}$$

where F is $m \times n$, with $n < m$, and of full rank?

[Hint: use a block LU factorization]

Solution: We start with

$$\begin{aligned}\begin{pmatrix} I & F \\ F^T & 0 \end{pmatrix} &= \begin{pmatrix} I & 0 \\ F^T & I \end{pmatrix} \begin{pmatrix} I & F \\ 0 & -F^T F \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ F^T & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -F^T F \end{pmatrix} \begin{pmatrix} I & F \\ 0 & I \end{pmatrix} \\ &= \begin{pmatrix} I & 0 \\ F^T & I \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -F^T F \end{pmatrix} \begin{pmatrix} I & 0 \\ F^T & I \end{pmatrix}^T\end{aligned}$$

This is of the form XDX^T where X is invertible.. Therefore the inertia is the same as that of the block diagonal matrix which is: m positive eigenvalues (block I) and n negative eigenvalues since $-F^T F$ is $n \times n$ and negative definite.