✍1 Non associativity in the presence of round-off.

Solution: This is done in a class demo and the diary should be posted. Here are the commands.

```
n = 10000;
a = randn(n,1);  b = randn(n,1);  c = randn(n,1);
t = ((a+b)+c == a+(b+c));
sum(t)
```

Right-hand side in 3rd line returns 1 for each instance when the two numbers are the same. □

✍2 Find machine epsilon in matlab.

Solution:

```
u = 1;
for i=0:999
    fprintf(1,' i =  %d , u = %e \n',i,u)
    if (1.0 +u == 1.0) break, end
```

```
    u = u/2;
end
u = u*2
```

□

✍4 Proof of Lemma: If $|\delta_i| \leq \underline{\mathbf{u}}$ and $n\underline{\mathbf{u}} < 1$ then

$$\Pi_{i=1}^{n}(1+\delta_i) = 1 + \theta_n \quad \text{where} \quad |\theta_n| \leq \frac{n\underline{\mathbf{u}}}{1 - n\underline{\mathbf{u}}}$$

Solution:

The proof is by induction on n.

1) Basis of induction. When $n = 1$ then the product reduces to $1 + \delta_i$ and so we can take $\theta_n = \delta_n$ and we know that $|\delta_n| \leq \underline{\mathbf{u}}$ from the assumptions and so

$$|\theta_n| \leq \underline{\mathbf{u}} \leq \frac{\underline{\mathbf{u}}}{1 - \underline{\mathbf{u}}},$$

as desired.

2) Induction step. Assume now that the result as stated is true for n and consider a product with $n+1$ terms: $\Pi_{i=1}^{n+1}(1+\delta_i)$. We can write this as $(1 + \delta_{n+1})\Pi_{i=1}^{n}(1 + \delta_i)$ and from the induction hypothesis

we get:

$$\Pi_{i=1}^{n+1}(1+\delta_i) = (1+\theta_n)(1+\delta_{n+1}) = 1+\theta_n+\delta_{n+1}+\theta_n\delta_{n+1}$$

with θ_n satisfying the inequality $\theta_n \le (n\underline{\mathbf{u}})/(1-n\underline{\mathbf{u}})$. We call θ_{n+1} the quantity $\theta_{n+1} = \theta_n + \delta_{n+1} + \theta_n\delta_{n+1}$, and we have

$$\begin{aligned}
|\theta_{n+1}| &= |\theta_n + \delta_{n+1} + \theta_n\delta_{n+1}| \\
&\le \frac{n\underline{\mathbf{u}}}{1-n\underline{\mathbf{u}}} + \underline{\mathbf{u}} + \frac{n\underline{\mathbf{u}}}{1-n\underline{\mathbf{u}}} \times \underline{\mathbf{u}} \\
&= \frac{n\underline{\mathbf{u}} + \underline{\mathbf{u}}(1-n\underline{\mathbf{u}}) + n\underline{\mathbf{u}}^2}{1-n\underline{\mathbf{u}}} = \frac{(n+1)\underline{\mathbf{u}}}{1-n\underline{\mathbf{u}}} \\
&\le \frac{(n+1)\underline{\mathbf{u}}}{1-(n+1)\underline{\mathbf{u}}}
\end{aligned}$$

This establishes the result with n replaced by $n+1$ as wanted and completes the proof. □

✍5 Assume you use single precision for which you have $\underline{\mathbf{u}} = 2. \times 10^{-6}$. What is the largest n for which $n\underline{\mathbf{u}} \le 0.01$ holds? Any conclusions for the use of single precision arithmetic?

Solution: We need $n \le 0.01/(2.0 \times 10^{-4})$ which gives $n \le 5,000$. Hence, single precision is inadequate for computations involving long inner products.

✍6 What does the main result on inner products imply for the case when $y = x$? [Contrast the relative accuracy you get in this case vs. the general case when $y \neq x$] $\square$

Solution: In this case we have

$$|fl(x^T x) - (x^T x)| \leq \gamma_n x^T x$$

which implies that we will always have a small relative error. Not true for the general case because the final result (forward form)

$$\left| fl(y^T x) - (y^T x) \right| \leq \gamma_n |y|^T |x|$$

does not imply a small relative error which would mean $|fl(y^T x) - (y^T x)| \leq \epsilon |y^T x|$ where ϵ is small. $\square$

✍7 Show for any x, y, there exist $\Delta x, \Delta y$ such that

$$\begin{aligned} fl(x^T y) &= (x + \Delta x)^T y, \quad \text{with} \quad |\Delta x| \leq \gamma_n |x| \\ fl(x^T y) &= x^T (y + \Delta y), \quad \text{with} \quad |\Delta y| \leq \gamma_n |y| \end{aligned}$$

Solution: The main result we proved is that

$$fl(y^T x) = \sum_{i=1}^{n} x_i y_i (1 + \theta_i) \qquad \text{where} \quad |\theta_i| \leq \gamma_n$$

The first relation comes from just attaching each $(1 + \theta_i)$ to x_i so x_i is replaced by $x_i + \theta_i x_i$ ... Similarly for the second relation. $\square$

✍8 (Continuation) Let A an $m \times n$ matrix, x an n-vector, and $y = Ax$. Show that there exist a matrix ΔA such

$$fl(y) = (A + \Delta A)x, \quad \text{with} \quad |\Delta A| \le \gamma_n |A|$$

Solution: The result comes from applying the result on inner products to each entry y_i of y – which is the inner product of row i with y. We use the first of the two results above:

$$fl(y_i) = (a_{i,:} + \Delta a_{i,:})^T y \quad \text{with} \quad |\Delta a_{i,:}| \le \gamma_n |a_{i,:}|$$

the result follows from expressing this in matrix form. □

✍9 (Continuation) From the above derive a result about a column of the product of two matrices A and B. Does a similar result hold for the product AB as a whole?

Solution: We can have a result for each column since this is just a matrix-vector product. However this does not translate into a result for AB because the ΔA we get for each column will depend on the column. Specifically, for the j-th column of B you will have a certain matrix $(\Delta A)_j$ such that $fl(AB(:,j)) = (A + (\Delta A)_j)B(:,j)$ with certain conditions as set in previous exercise. However this $(\Delta A)_j$ is *NOT* the same matrix for each column. So you cannot say $fl(A) = (A + \Delta A)B$, ... $\square$

Supplemental notes

The importance of floating point analysis cannot be overstated. There were many instances where poor implementation of algorithms failed and led to - on occasion - disastrous results. One of the best examples is the failed launch of the European Ariane rocket in 1996 [Ariane flight V88]. See the story in this wikipedia page

https://en.wikipedia.org/wiki/Ariane_flight_V88