✍1 Show that $\kappa(I) = 1$;

Solution: This is obvious because for any matrix norm $\|I\| = \|I^{-1}\| = 1$. $\square$

✍2 Show that $\kappa(A) \geq 1$;

Solution: We have $\|AA^{-1}\| = \|I\| = 1$ therefore $1 = \|AA^{-1}\| \leq \|A\| \, \|A^{-1}\| = \kappa(A)$ $\square$

✍5 Show that if $\|E\|/\|A\| \leq \delta$ and $\|e_b\|/\|b\| \leq \delta$ then

$$\frac{\|x - y\|}{\|x\|} \leq \frac{2\delta\kappa(A)}{1 - \delta\kappa(A)}$$

Solution: From the main theorem (theorem 1) we have

$$\frac{\|x - y\|}{\|x\|} \leq \frac{\|A^{-1}\| \, \|A\|}{1 - \|A^{-1}\| \, \|E\|} \left(\frac{\|E\|}{\|A\|} + \frac{\|e_b\|}{\|b\|} \right)$$

If $\|E\| \leq \delta$ and $\|e_b\|/\|b\| \leq \delta$ then:

$$\frac{\|x-y\|}{\|x\|} \leq \frac{\kappa(A) \times 2\delta}{1 - \|A^{-1}\| \, \|E\|}$$
$$\leq \frac{2\delta\kappa(A)}{1 - \|A^{-1}\|\|A\| \times (\|E\|/\|A\|)}$$
$$\leq \frac{2\delta\kappa(A)}{1 - \delta\kappa(A)}.$$

$\square$

✍9 Show that $\frac{\|x-\tilde{x}\|}{\|x\|} \geq \frac{1}{\kappa(A)} \frac{\|r\|}{\|b\|}$.

Solution: We start with noting that $A(x - \tilde{x}) = b - A\tilde{x} = r$. So:

$$\|r\| \leq \|A\|\|x - \tilde{x}\| \to \frac{\|r\|}{\|b\|} \leq \|A\| \frac{\|x - \tilde{x}\|}{\|b\|}$$

Next from $\|x\| = \|A^{-1}b\| \leq \|A^{-1}\|\|b\|$ we get $\|b\| \geq \|x\|/\|A^{-1}\|$ and so

$$\frac{\|r\|}{\|b\|} \leq \|A\| \frac{\|x - \tilde{x}\|}{\|x\|/\|A^{-1}\|} = \kappa(A) \frac{\|x - \tilde{x}\|}{\|x\|}$$

which yields the result after dividing the 2 sides by $\kappa(A)$. $\square$

More on normwise backward error

➤ We solve $Ax = b$ and find an approximate solution y

Question: Find smallest perturbation to apply to A, b so that *exact* solution of perturbed system is y

➤ We now consider the situation where only b or A are perturbed (not both)

Suppose we model entire perturbation in RHS b.

➤ Let $r = b - Ay$ be the residual. Then y satisfies $Ay = b + \Delta b$ with $\Delta b = -r$ exactly.

➤ The relative perturbation to the RHS is $\frac{\|r\|}{\|b\|}$.

Suppose we model entire perturbation in matrix A.

➤ Then y satisfies $\left(A + \frac{ry^T}{y^Ty}\right) y = b$

➤ The relative perturbation to the matrix is

$$\left\|\frac{ry^T}{y^Ty}\right\|_2 / \|A\|_2 = \frac{\|r\|_2}{\|A\| \, \|y\|_2}$$

Proof of Theorem 3

Let $D \equiv \|E\|\|y\| + \|e_b\|$ and $\eta \equiv \eta_{E,e_b}(y)$. The theorem states that $\eta = \|r\|/D$ (recall that $r = b - Ay$). Proof in 2 steps.

First: Any $\Delta A, \Delta b$ pair satisfying (1) is such that $\epsilon \geq \|r\|/D$. Indeed from (1) we have:

$$Ay + \Delta Ay = b + \Delta b \to r = \Delta Ay - \Delta b \to$$
$$\|r\| \leq \|\Delta A\|\|y\| + \|\Delta b\|$$
$$\leq \epsilon(\|E\|\|y\| + \|e_b\|) \to$$
$$\epsilon \geq \frac{\|r\|}{D}$$

Second: We need to show an instance where the minimum value of $\|r\|/D$ is reached. Take the pair $\Delta A, \Delta b$:

$$\Delta A = \alpha r z^T; \; \Delta b = \beta r \text{ with } \alpha = \frac{\|E\|\|y\|}{D}; \; \beta = -\frac{\|e_b\|}{D}$$

The vector z depends on the norm used - for the 2-norm: $z = y/\|y\|^2$. Here: Proof only for 2-norm

Next, we need to verify that first part of (1) is satisfied:

$$\begin{aligned}(A+\Delta A)y &= Ay + \alpha r \frac{y^T}{\|y\|^2} y = b - r + \alpha r \\ &= b - (1-\alpha) r \\ &= b - \left(1 - \frac{\|E\|\,\|y\|}{\|E\|\,\|y\| + \|e_b\|}\right) r \\ &= b - \frac{\|e_b\|}{D} r = b + \beta r \quad \rightarrow \\ (A+\Delta A)y &= b + \Delta b \quad \leftarrow \text{The desired result}\end{aligned}$$

Finally: Must now verify that $\|\Delta A\| = \eta\|E\|$ and $\|\Delta b\| = \eta\|e_b\|$. Exercise: Show that $\|uv^T\|_2 = \|u\|_2\|v\|_2$

$$\|\Delta A\| = \frac{|\alpha|}{\|y\|^2}\|ry^T\| = \frac{\|E\|\,\|y\|}{D}\frac{\|r\|\,\|y\|}{\|y\|^2} = \eta\|E\|$$

$$\|\Delta b\| = |\beta|\|r\| = \frac{\|e_b\|}{D}\|r\| = \eta\|e_b\| \qquad QED$$