✍1 Show that each $A_k$ [$A(1:k,1:k)$ in matlab notation] is SPD.

**Solution:** Let $x$ be any vector in $\mathbb{R}^k$ and consider the vector $y$ of $\mathbb{R}^n$ obtained by stacking $x$ followed by $n-k$ zeros. Then it can be easily seen that : $(A_k x, x) = (Ay, y)$ and since $A$ is SPD then $(Ay, y) > 0$ and therefore $(A_k x, x) > 0$ for any $x$ in $\mathbb{R}^k$. Hence $A_k$ is SPD. □

✍2 Consequence $\det(A_k) > 0$

**Solution:** This is because the determinant is the product of the eigenvalues which are real positive (see notes). □

✍3 If $A$ is SPD then for any $n \times k$ matrix $X$ of rank $k$, the matrix $X^T AX$ is SPD.

**Solution:** For any $v \in \mathbb{R}^k$ we have $(X^T AXv, v) = (AXv, Xv)$. In addition, since $X$ is of full rank, then $Xv$ cannot be zero if $v$ is nonzero. Therefore we have $(AXv, Xv) > 0$. □

✍4 Show that if $A^T = A$ and $(Ax, x) = 0 \; \forall x$ then $A = 0$.

**Solution:** The condition implies that for all $x, y$ we have $(A(x + y), x + y) = 0$. Now expand this as: $(Ax, x) + (Ay, y) + 2(Ax, y) = 0$ for all $x, y$ which shows that $(Ax, y) = 0 \; \forall \; x, y$. This implies that $A = 0$ (e.g. take $x = e_j, y = e_i$)... $\square$

✍5 Show: A nonzero matrix $A$ is indefinite iff:

$$\exists \; x, y : (Ax, x)(Ay, y) < 0.$$

**Solution:**

[$\leftarrow$] Trivial. The matrix cant be PSD or NSD under the conditon

[$\rightarrow$] Need to prove: If $A$ is indefinite then there exist such that $x, y : (Ax, x)(Ay, y) < 0$. Assume contrary is true, i.e.,

$$\forall \; x, y (Ax, x)(Ay, y) \geq 0$$

. There is at least one $x_0$ such that $(Ax_0, x_0)$ is nonzero, otherwise $A = 0$ from previous question. Assume $(Ax_0, x_0) > 0$. Then $\forall y (Ax_0, x_0)(Ay, y) \geq 0$. which implies $\forall y : (Ay, y) \geq 0$, i.e., $A$ is positive semi-definite. This contradicts the assumption that $A$ is neither positive nor negative semi-defininte $\square$

✍6 The (standard) LU factorization of an SPD matrix $A$ exists.

**Solution:** This is an immediate consequence of the main theorem on existence (Lec. notes. set #5) and Exercise 1 in this set which showed that $\det(A_k) > 0$ for $k = 1, \cdots, n$. $\square$

***Example:***

$$A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 5 & 0 \\ 2 & 0 & 9 \end{pmatrix}$$

✍7 Is $A$ symmetric positive definite?

**Solution:** Answer is yes because $\det(A_k) > 0$ for $k = 1, 2, 3$. $\square$

✍8 What is the $LDL^T$ factorization of $A$ ?

**Solution:** The LU factorizatis is:

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 1/2 & 1 \end{pmatrix} \qquad U = \begin{pmatrix} 1 & -2 & 1 \\ 0 & 4 & 2 \\ 0 & 0 & 4 \end{pmatrix}$$

Therefore $A = LDL^T$ where $L$ is as given above and

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix} \qquad \square$$

✍9 What is the Cholesky factorization of $A$ ?

**Solution:** From the above LDLT factorization we have $A = GG^T$ with

$$G = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 2 & 0 \\ 2 & 1 & 2 \end{pmatrix} \qquad \square$$

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## Gradient of $\psi(x) = (Ax, x)$

In practice exercise # 6 it is asked: Let $A$ be symmetric and $\psi(x) = (Ax, x)$. What is the partial derivative $\frac{\partial \psi(x)}{\partial x_k}$? What is the gradient of $\psi$?

**Solution:** First note that

$$\psi(x) = \sum_{i=1}^{n} x_i \left[ \sum_{j=1}^{n} a_{ij} x_j \right]$$

and so, using basic rules for derivatives of products:

$$\begin{aligned}
\frac{\partial \psi(x)}{\partial x_k} &= \sum_{i=1}^{n} \frac{\partial x_i}{\partial x_k} \left[ \sum_{j=1}^{n} a_{ij} x_j \right] + \sum_{i=1}^{n} x_i \left[ \frac{\partial x_i}{\partial x_k} \sum_{j=1}^{n} a_{ij} x_j \right] \\
&= \sum_{j=1}^{n} a_{kj} x_j + \sum_{i=1}^{n} x_i a_{ik} \\
&= 2 \sum_{j=1}^{n} a_{kj} x_j
\end{aligned}$$

which is nothing but twice the $k$-th component of $Ax$ or $\frac{\partial \psi(x)}{\partial x_k} = 2(Ax)_k$. Therefore the gradient of $\psi$ is

$$\nabla \psi(x) = 2Ax.$$

A somewhat simpler solution for finding the gradient is to expand $\psi(x + \delta) = (A(x + \delta), (x + \delta)) = \ldots$ and to write that the linear term should be of the form $[\nabla\psi]^T\,\boldsymbol{\delta}$. $\square$

*Supplemental notes on Column Cholesky.* Let $A = GG^T$ with $G$ = lower triangular. Then equate $j$-th columns:

$$a(:,j) = \sum_{k=1}^{j} g(:,k) g^T(k,j) \rightarrow$$

$$\begin{aligned} A(:,j) &= \sum_{k=1}^{j} G(j,k) G(:,k) \\ &= G(j,j)G(:,j) + \sum_{k=1}^{j-1} G(j,k)G(:,k) \rightarrow \\ G(j,j)G(:,j) &= A(:,j) - \sum_{k=1}^{j-1} G(j,k)G(:,k) \end{aligned}$$

- Assume that first $j-1$ columns of $G$ already known.
- Compute unscaled column-vector:

$$v = A(:,j) - \sum_{k=1}^{j-1} G(j,k)G(:,k)$$

- Notice that $v(j) \equiv G(j,j)^2$.
- Compute $\sqrt{v(j)}$ and scale $v$ to get $j$-th column of $G$.

## ALGORITHM : 1 ■ *Column Cholesky*

1. *For* $j = 1 : n$ *do*
2. *For* $k = 1 : j - 1$ *do*
3. $A(j : n, j) = A(j : n, j) - A(j, k) * A(j : n, k)$
4. *EndDo*
5. *If* $A(j, j) \leq 0$ *ExitError("Matrix not SPD")*
6. $A(j, j) = \sqrt{A(j, j)}$
7. $A(j + 1 : n, j) = A(j + 1 : n, j)/A(j, j)$
8. *EndDo*