✍2 Prove that Gram-Schmidt can be completed iff the x_i's are linearly independent.

Solution:

We will show that Gram-Schmidt breaks down iff the x_i's are linearly dependent.

The only way in which GS can break down is if $r_{jj} = \|\hat{q}\|_2$ in line 7 is zero.

The main observation is that the vector $\hat{q}$ at the end of the loop starting in line 4 (ending in line 6) is a linear combination of the x_i's for $i = 1, \cdots, j$. [simple proof by induction – omitted.] □

✍3 Cost of Gram-Schmidt?

Solution: Step j of the algorithm costs : $(j-1) \times 2m$ operations for line 3, + $(j-1) \times 2m$ operations for loop in line 4 + $3m$ operations in Lines 7 and 8 together. Total for step j = $c_j = (4j-1)m$. Total over the n columns = $T(n) = (2n^2 + n)m \approx 2n^2m$.

Note: this is linear in m (number of rows) and quadratic in n (number of columns). $\square$

✍5 What is the cost of solving a linear system with the QR factorization?

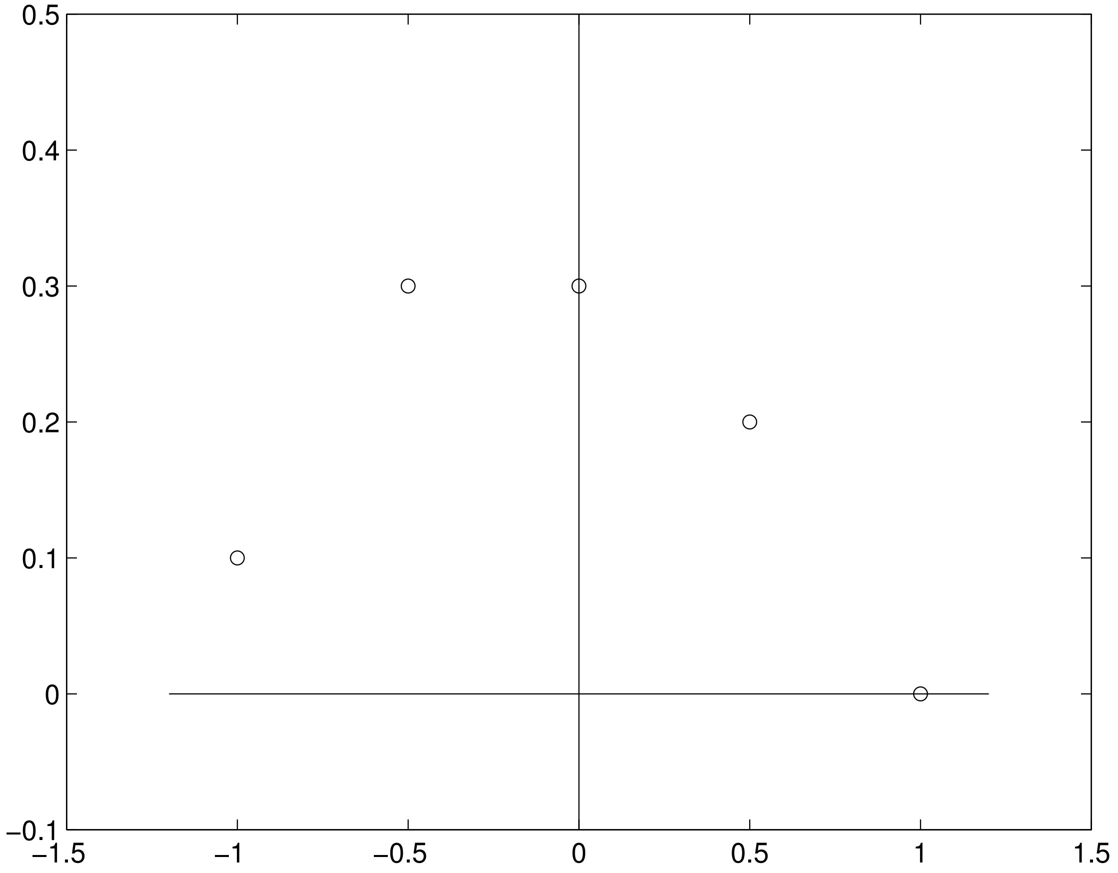
Solution: According to the previous question we have a cost of $2n^3$ for the factorization (since $m = n$), to which we need to add the cost of solving a triangular solve $O(n^2)$ and the cost for computing $Q^T b$ which is again $O(n^2)$. In the end the cost is dominated by the QR factorization which is $2n^3$. This is 3 times more expensive than GE. $\square$

Supplementatal notes: Examples

➤ Data fitting

Example:

t_i:	-1	$-1/2$	0	$1/2$	1
β_i:	0.1	0.3	0.3	0.2	0.0



1) Approximations by polynomials of degree one:

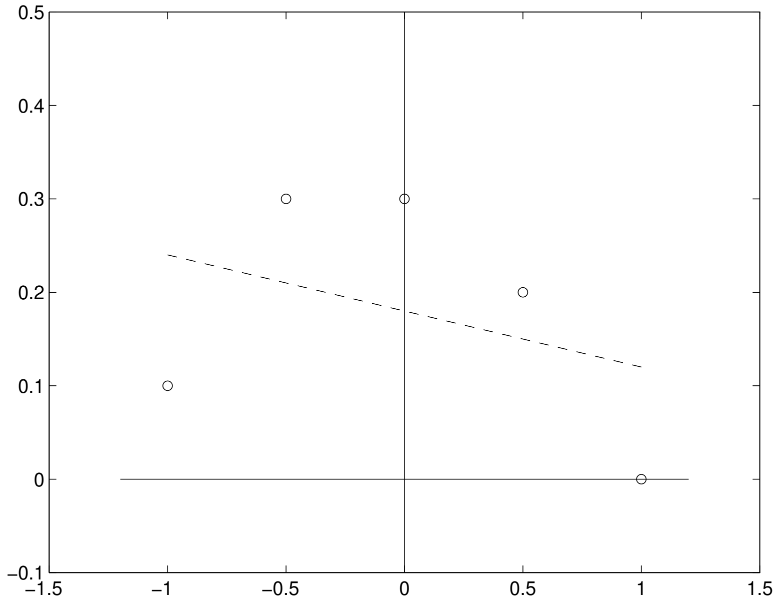
- $\phi_1(t) = 1, \phi_2(t) = t.$

$$F = \begin{pmatrix} 1.0 & -1.0 \\ 1.0 & -0.5 \\ 1.0 & 0 \\ 1.0 & 0.5 \\ 1.0 & 1.0 \end{pmatrix}$$

$$F^T F = \begin{pmatrix} 5.0 & 0 \\ 0 & 2.5 \end{pmatrix}$$

$$F^T b = \begin{pmatrix} 0.9 \\ -0.15 \end{pmatrix}$$

- Best approximation is $\boxed{\phi(t) = 0.18 - 0.06t}$.



2) Approximation by polynomials of degree 2:

- $\phi_1(t) = 1, \phi_2(t) = t, \phi_3(t) = t^2.$
- Best polynomial found:

$$0.30857.. - 0.06 \times t - 0.25715... \times t^2$$