SOLVING LINEAR SYSTEMS OF EQUATIONS

- Quick background on linear systems
- The Gaussian elimination algorithm (review)
- The LU factorization
- Gaussian Elimination with pivoting – permutation matrices.
- Case of banded systems

Background: Linear systems

The Problem: $\boldsymbol{A}$ is an $\boldsymbol{n} \times \boldsymbol{n}$ matrix, and $\boldsymbol{b}$ a vector of $\mathbb{R}^n$. Find $\boldsymbol{x}$ such that:

$$\boldsymbol{Ax = b}$$

➤ $\boldsymbol{x}$ is the unknown vector, $\boldsymbol{b}$ the right-hand side, and $\boldsymbol{A}$ is the coefficient matrix

Example:

$$\left\{\begin{array}{rcrcrcl} 2x_1 & + & 4x_2 & + & 4x_3 & = & 6 \\ x_1 & + & 5x_2 & + & 6x_3 & = & 4 \\ x_1 & + & 3x_2 & + & x_3 & = & 8 \end{array}\right. \quad \text{or} \quad \begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 6 \\ 4 \\ 8 \end{pmatrix}$$

✍1 Solution of above system ?

- Standard mathematical solution by Cramer's rule:

$$\boxed{x_i = \det(A_i)/\det(A)}$$

$A_i =$ matrix obtained by replacing i-th column by b.

- Note: This formula is useless in practice beyond $n = 3$ or $n = 4$.

Three situations:

1. The matrix A is nonsingular. There is a unique solution given by $x = A^{-1}b$.
2. The matrix A is singular and $b \in \mathrm{Ran}(A)$. There are infinitely many solutions.
3. The matrix A is singular and $b \notin \mathrm{Ran}(A)$. There are no solutions.

Example: (1) Let $A = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 8 \end{pmatrix}$. A is nonsingular ➤ a unique solution $x = \begin{pmatrix} 0.5 \\ 2 \end{pmatrix}$.

Example: (2) Case where A is singular & $b \in \text{Ran}(A)$:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

➤ infinitely many solutions: $x(\alpha) = \begin{pmatrix} 0.5 \\ \alpha \end{pmatrix} \quad \forall \alpha.$

Example: (3) Let A same as above, but $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

➤ No solutions since 2nd equation cannot be satisfied

Background. Triangular linear systems

Example:

$$\begin{pmatrix} 2 & 4 & 4 \\ 0 & 5 & -2 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix}$$

- One equation can be trivially solved: the last one. $x_3 = 2$
- x_3 is known we can now solve the 2nd equation:

$$5x_2 - 2x_3 = 1 \rightarrow 5x_2 - 2 \times 2 = 1 \rightarrow x_2 = 1$$

- Finally x_1 can be determined similarly:

$$2x_1 + 4x_2 + 4x_3 = 2 \rightarrow \ldots \rightarrow x_1 = -5$$

ALGORITHM : 1 . *Back-Substitution algorithm*

```
For $i = n : -1 : 1$ do:
    $t := b_i$
    For $j = i + 1 : n$ do
        $t := t - a_{ij}x_j$
    End
    $x_i = t/a_{ii}$
End
```

$$\left.\begin{array}{l}\text{For } j = i+1:n \text{ do}\\ \quad t := t - a_{ij}x_j\\ \text{End}\end{array}\right\} \begin{aligned} t &:= b_i - (a_{i,i+1:n}, x_{i+1:n}) \\ &= b_i - \textit{an inner product}\end{aligned}$$

- We must require that each $a_{ii} \neq 0$
- Operation count?

Column version of back-substitution

Back-Substitution algorithm. Column version

```
For $j = n : -1 : 1$ do:
    $x_j = b_j / a_{jj}$
    For $i = 1 : j - 1$ do
        $b_i := b_i - x_j * a_{ij}$
    End
End
```

✍2 Justify the above algorithm [Show that it does indeed compute the solution]

➤ Analogous algorithms for *lower* triangular systems.

Background: Gaussian Elimination

- Back to arbitrary linear systems.

Principle of the method: Since triangular systems are easy to solve, we will transform a linear system into one that is triangular. Main operation: combine rows so that zeros appear in the required locations to make the system triangular.

Notation: use a Tableau:

$$\left\{\begin{array}{rcrcrcr} 2x_1 & + & 4x_2 & + & 4x_3 & = & 2 \\ x_1 & + & 3x_2 & + & 1x_3 & = & 1 \\ x_1 & + & 5x_2 & + & 6x_3 & = & -6 \end{array}\right. \quad \text{tableau:} \quad \begin{array}{|ccc|c|} \hline 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \\ \hline \end{array}$$

- Main operation used: scaling and adding rows.

Example: Replace row2 by: row2 - $\frac{1}{2}$*row1:

$$\left|\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right| \rightarrow \left|\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 5 & 6 & -6 \end{array}\right|$$

- This is equivalent to:

$$\left|\begin{array}{ccc} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ 0 & 0 & 1 \end{array}\right| \times \left|\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right| = \left|\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 5 & 6 & -6 \end{array}\right|$$

- The left-hand matrix is of the form $M = I - ve_1^T$ with $v = \begin{pmatrix} 0 \\ \frac{1}{2} \\ 0 \end{pmatrix}$

Gaussian Elimination (cont.)

Go back to original system. Step 1 must transform:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right] \text{ into: } \left[\begin{array}{ccc|c} x & x & x & x \\ 0 & x & x & x \\ 0 & x & x & x \end{array}\right]$$

$row_2 := row_2 - \frac{1}{2} \times row_1$: $\quad$ $row_3 := row_3 - \frac{1}{2} \times row_1$:

$$\left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 1 & 5 & 6 & -6 \end{array}\right] \qquad \left[\begin{array}{ccc|c} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right]$$

- Equivalent to

$$\left|\begin{array}{rrr} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 1 \end{array}\right| \times \left|\begin{array}{rrr|r} 2 & 4 & 4 & 2 \\ 1 & 3 & 1 & 1 \\ 1 & 5 & 6 & -6 \end{array}\right| = \left|\begin{array}{rrr|r} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right|$$

$$[\boldsymbol{A},\boldsymbol{b}] \rightarrow [\boldsymbol{M}_1\boldsymbol{A},\boldsymbol{M}_1\boldsymbol{b}]; \quad \boldsymbol{M}_1 = \boldsymbol{I} - \boldsymbol{v}^{(1)}\boldsymbol{e}_1^T; \quad \boldsymbol{v}^{(1)} = \begin{pmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}$$

- New system $\boldsymbol{A}_1\boldsymbol{x} = \boldsymbol{b}_1$. Step 2 must now transform:

$$\left|\begin{array}{rrr|r} 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \end{array}\right| \text{ into: } \left|\begin{array}{ccc|c} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \end{array}\right|$$

$$row_3 := row_3 - 3 \times row_2 : \rightarrow \begin{array}{|rrr|r|}\hline 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \\ \hline \end{array}$$

- Equivalent to

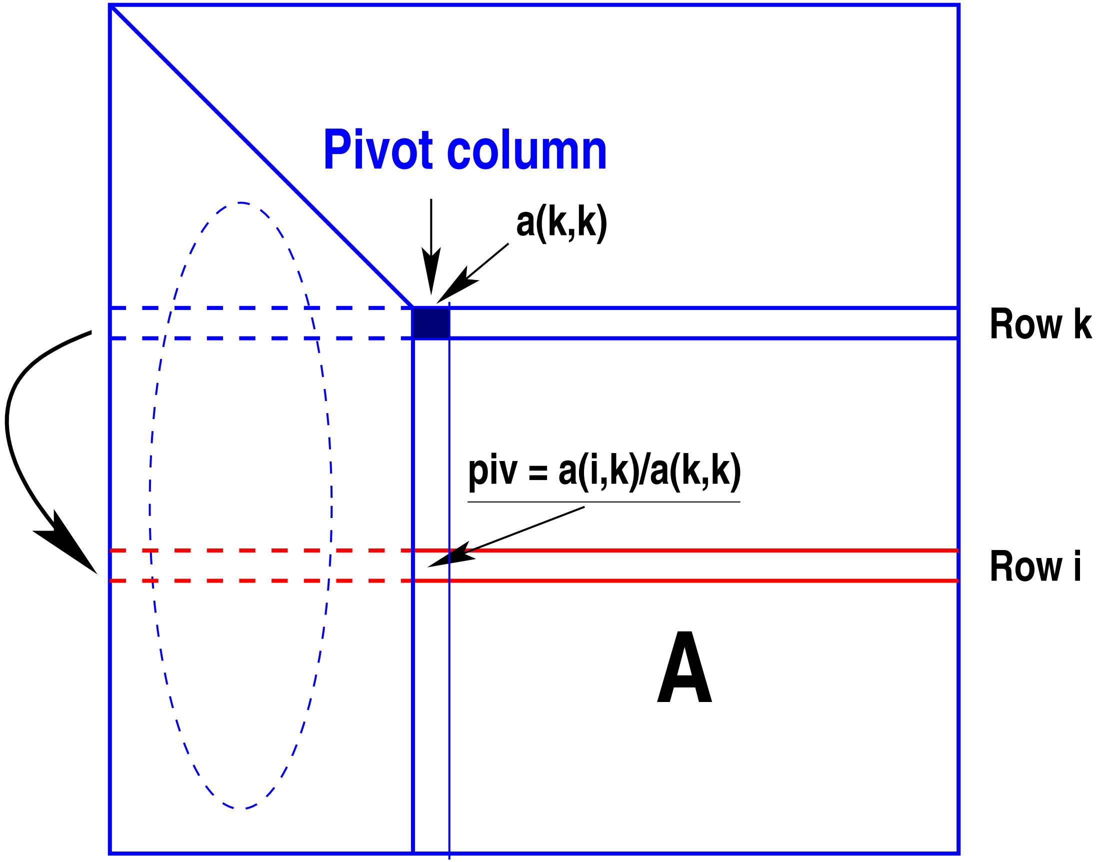
$$\begin{array}{|rrr|}\hline 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \\ \hline \end{array} \times \begin{array}{|rrr|r|}\hline 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & 4 & -7 \\ \hline \end{array} = \begin{array}{|rrr|r|}\hline 2 & 4 & 4 & 2 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 7 & -7 \\ \hline \end{array}$$

- Second transformation is as follows:

$$[A_1, b_1] \rightarrow [M_2 A_1, M_2 b_1]; \; M_2 = I - v^{(2)} e_2^T; \; v^{(2)} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

- Triangular system ➤ Solve.

Gaussian Elimination in a picture



```
For i=k+1:n Do:
    piv = a(i,k)/a(k,k)
    row(i):=row(i) – piv*row(k)
```

ALGORITHM : 2 ▪ Gaussian Elimination

```
For $k = 1 : n-1$ Do:
    For $i = k+1 : n$ Do:
        $piv := a_{ik}/a_{kk}$
        For $j := k+1 : n+1$ Do :
            $a_{ij} := a_{ij} - piv * a_{kj}$
        End
    End
End
```

➤ Operation count:

$$T = \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} [1 + \sum_{j=k+1}^{n+1} 2] = \sum_{k=1}^{n-1} \sum_{i=k+1}^{n} (2(n-k)+3) = \dots$$

✍3 Complete the above calculation. Order of the cost?

The LU factorization

➤ Now ignore the right-hand side and consider only A

Observation: Gaussian elimination is equivalent to $n-1$ successive Gaussian transformations, i.e., multiplications with matrices of the form $M_k = I - v^{(k)} e_k^T$, where the first k components of $v^{(k)}$ equal zero.

$$\begin{aligned} A_1 &= M_1 A_0 \quad \text{with} \quad A_0 \equiv A \\ A_2 &= M_2 A_1 = M_2(M_1 A_0) \\ A_3 &= M_3 A_2 = M_3(M_2 M_1 A_0) \\ \cdots &= \cdots \\ A_{n-1} &= M_{n-1} \cdots M_1 A_0 \rightarrow \\ U &= [M_{n-1} \cdots M_1] A \end{aligned}$$

$$A = \underbrace{[M_{n-1}M_{n-2}\ldots M_1]^{-1}}_{L} U \equiv LU$$

LU decomposition : $A = LU$, where L is lower triangular with ones on diagonal ('Unit lower triang.), and U is upper triangular = the last matrix obtained in the process $= (A_{n-1})$.

➤ Easy to get U. How do we get L? Can show:

The L factor is a lower triangular matrix with ones on the diagonal. Column k of L, contains the multipliers l_{ik} used in the k-th step of Gaussian elimination.

➤ There is an 'algorithmic' approach to understanding the LU factorization [see supplemental notes]

A matrix $\boldsymbol{A}$ has an LU decomposition iff

$$\det(A(1:k,1:k)) \neq 0 \quad \text{for} \quad k = 1, \cdots, n-1.$$

In this case: $\det A = \det(U) = \prod_{i=1}^{n} u_{ii}$

If, in addition, $\boldsymbol{A}$ is nonsingular, then the LU factorization is unique.

✍4 Practical use: Show how to use the LU factorization to solve linear systems with the same matrix $\boldsymbol{A}$ and different $\boldsymbol{b}$'s.

✍5 LU factorization of the matrix $A = \begin{pmatrix} 2 & 4 & 4 \\ 1 & 5 & 6 \\ 1 & 3 & 1 \end{pmatrix}$? ✍6 Determinant of $\boldsymbol{A}$?

✍7 True or false: "Computing the LU factorization of matrix $\boldsymbol{A}$ involves more arithmetic operations than solving a linear system $Ax = b$ by Gaussian elimination".

Gaussian Elimination: Partial Pivoting

Consider again GE for the system:

$$\begin{cases} 2x_1 + 2x_2 + 4x_3 = 2 \\ x_1 + x_2 + x_3 = 1 \\ x_1 + 4x_2 + 6x_3 = -5 \end{cases} \quad \text{Or:} \quad \begin{array}{|ccc|c|} \hline 2 & 2 & 4 & 2 \\ 1 & 1 & 1 & 1 \\ 1 & 4 & 6 & -5 \\ \hline \end{array}$$

- $row_2 := row_2 - \frac{1}{2} \times row_1$:

$$\begin{array}{|ccc|c|} \hline 2 & 2 & 4 & 2 \\ 0 & 0 & -1 & 0 \\ 1 & 4 & 6 & -5 \\ \hline \end{array}$$

- $row_3 := row_3 - \frac{1}{2} \times row_1$:

$$\begin{array}{|ccc|c|} \hline 2 & 2 & 4 & 2 \\ 0 & 0 & -1 & 0 \\ 0 & 3 & 4 & -6 \\ \hline \end{array}$$

- Pivot a_{22} is zero. Solution : permute rows 2 and 3:

$$\begin{array}{|ccc|c|} \hline 2 & 2 & 4 & 2 \\ 0 & 3 & 4 & -6 \\ 0 & 0 & -1 & 0 \\ \hline \end{array}$$

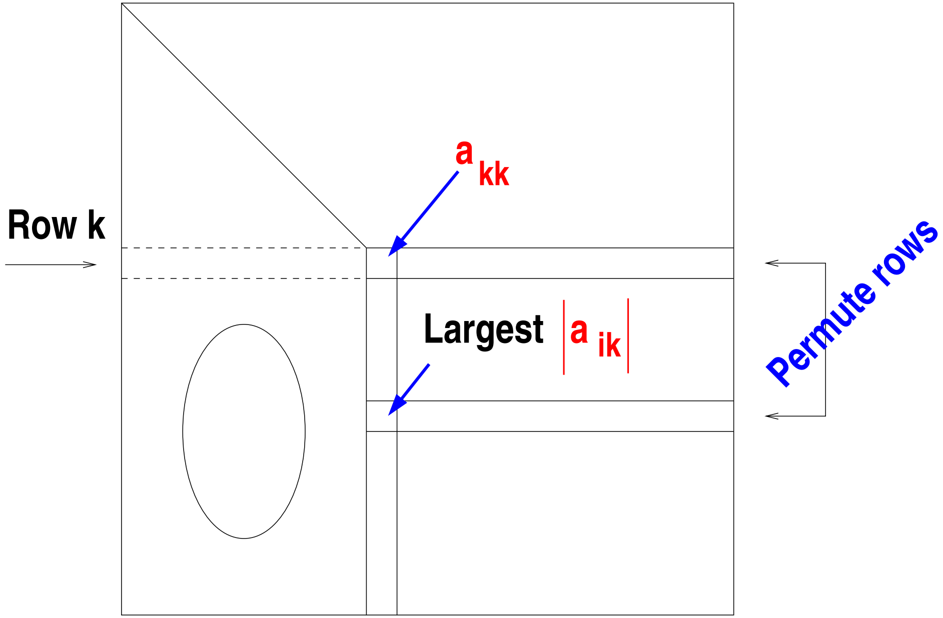
Gaussian Elimination with Partial Pivoting

Partial Pivoting

- General step k shown on the right $\longrightarrow$
- Exchange row k with row l where

$$|a_{lk}| = \max_{i=k,\ldots,n} |a_{ik}|$$

- Do this at each step
- Yields a more 'stable' algorithm.



✍8 The matlab script *gaussp* will be provided. Explore it from the angle of an actual implementation in a language like C. Is it necessary to 'physically' move the rows? (moving data around is not free).

Pivoting and permutation matrices

➤ A permutation matrix is a matrix obtained from the identity matrix by permuting its rows

➤ For example for the permutation $\pi = \{3, 1, 4, 2\}$ we obtain

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

➤ Important observation: the matrix PA is obtained from A by permuting its rows with the permutation π

$$(PA)_{i,:} = A_{\pi(i),:}$$

✍9 What is the matrix $\boldsymbol{PA}$ when

$$\boldsymbol{P} = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} \end{pmatrix} \quad \boldsymbol{A} = \begin{pmatrix} \mathbf{1} & \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{5} & \mathbf{6} & \mathbf{7} & \mathbf{8} \\ \mathbf{9} & \mathbf{0} & -\mathbf{1} & \mathbf{2} \\ -\mathbf{3} & \mathbf{4} & -\mathbf{5} & \mathbf{6} \end{pmatrix} ?$$

➤ Any permutation matrix is the product of interchange permutations, which only swap two rows of $\boldsymbol{I}$.

➤ Notation: $\boldsymbol{E}_{ij}$ = Identity with rows $\boldsymbol{i}$ and $\boldsymbol{j}$ swapped

Example: To obtain $\pi = \{3, 1, 4, 2\}$ from $\pi = \{1, 2, 3, 4\}$ – we need to swap $\pi(2) \leftrightarrow \pi(3)$ then $\pi(3) \leftrightarrow \pi(4)$ and finally $\pi(1) \leftrightarrow \pi(2)$. Hence:

$$P = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix} = E_{1,2} \times E_{3,4} \times E_{2,3}$$

✍10 In the previous example where

```
>> A = [ 1  2  3  4;  5  6  7  8;  9   0 -1  2 ;  -3  4  -5  6]
```

Matlab gives $det(A) = -896$. What is $det(PA)$?

Obtaining the LU factorization with pivoting

➤ The main result is simple (though cumbersome to prove)

We end up with the factorization

$$\boxed{PA = LU}$$

where

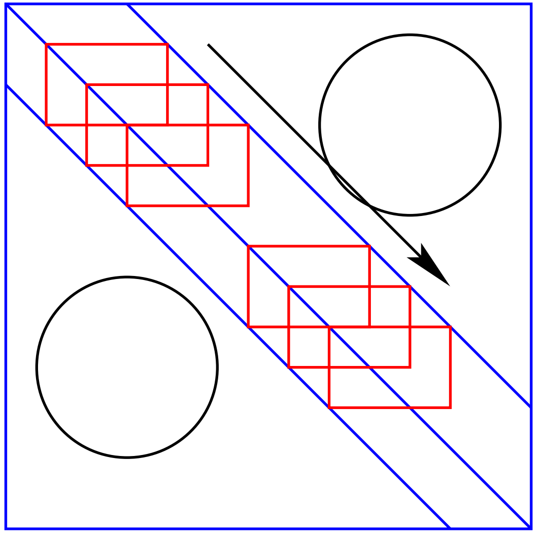
- P is the permutation matrix corresponding to the accumulated swaps.
- U is the last upper triangular matrix obtained
- L is the same matrix of multipliers as before, *but* the rows are swapped when those of the (evolving) U are.

➤ Best explained with examples.

Special case of banded matrices

- Banded matrices arise in many applications
- A has upper bandwidth q if $a_{ij} = 0 \text{ for } j - i > q$
- A has lower bandwidth p if $a_{ij} = 0 \text{ for } i - j > p$

✍11 Explain how GE would work on a banded system (you want to avoid operations involving zeros) – Hint: see picture



- Simplest case: tridiagonal ➤ $p = q = 1.$

➤ First observation: Gaussian elimination (no pivoting) preserves the initial banded form. Consider first step of Gaussian elimination:

```
For $i = 2 : n$ Do:
    $a_{i1} := a_{i1}/a_{11}$ (pivots)
    For $j := 2 : n$ Do :
        $a_{ij} := a_{ij} - a_{i1} * a_{1j}$
    End
End
```

➤ If A has upper bandwidth q and lower bandwidth p then so is the resulting $[L/U]$ matrix. ➤ Band form is preserved (induction)

✍12 Operation count?

What happens when partial pivoting is used?

If A has lower bandwidth p, upper bandwidth q, and if Gaussian elimination with partial pivoting is used, then the resulting U has upper bandwidth $p+q$. L has at most $p+1$ nonzero elements per column (bandedness is lost).

- Simplest case: tridiagonal ➤ $p = q = 1$.

Example:

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 1 & 0 \\ 0 & 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$