Math 5652: Introduction to Stochastic Processes: Spring 2014

Appendix A. Generating functions

Let X be a random variable with values in $\mathbb{R}^1$. The **generating**, or **probability generating function** of X is defined as $\phi(t) = \phi_X(t) = E(t^X)$. If X has discrete distribution with $p_k = P(X = k)$, $k = 0, 1, 2, \ldots$, then

$$\phi(t) = \sum_{k=0}^{\infty} p_k t^k \quad \text{for} \quad |t| \le 1, \quad \text{and} \quad p_k = \frac{\phi^{(k)}(0)}{k!}.$$

In this case, we also have

$$\phi(1) = 1, \qquad \phi^{(k)}(1) = E\big(X(X-1)\cdots(X-k-1)\big) \quad \text{for} \quad k = 1, 2, \ldots.$$

The **moment generating function** of X is $\varphi(t) = \varphi_X(t) = E\big(e^{tX}\big) = \phi(e^t)$. If it is defined in a neighborhood of the point $t = 0$, then the k^{th} moment of X,

$$E(X^k) = \varphi^{(k)}(0), \qquad k = 1, 2, \ldots.$$

In this case, we also have

$$E(X) = \psi'(0), \quad \mathrm{Var}\,(X) = \psi''(0), \qquad \text{where} \quad \psi(t) = \ln \varphi(t).$$

Note that if $X_1, X_2, \ldots, X_n$ are independent, and $X = X_1 + X_2 + \cdots + X_n$, then

$$\varphi_X = \varphi_{X_1} \cdot \varphi_{X_2} \cdot \ldots \cdot \varphi_{X_n}, \qquad \psi_X = \psi_{X_1} + \psi_{X_2} + \ldots + \psi_{X_n}.$$

1. $X =$ **Binomial** (n, p) with $n = 1, 2, \ldots;\ 0 \le p \le 1$.

$$f_1(k) = P(X = k) = \binom{n}{k} p^k q^{n-r} \quad \text{for} \quad k = 0, 1, \ldots, n; \quad \text{where} \quad q = 1 - p;$$

$$\phi_1(t) = E(t^X) = (pt + q)^n, \quad \varphi_1(t) = E\big(e^{tX}\big) = \big(p\,e^t + q\big)^n, \quad \mu_1 = E(X) = np, \quad \sigma_1^2 = \mathrm{Var}\,(X) = npq.$$

2. $X =$ **Poisson** (λ) with $\lambda > 0$.

$$f_2(k) = P(X = k) = e^{-\lambda} \cdot \frac{\lambda^k}{k!} \quad \text{for} \quad k = 0, 1, 2 \ldots;$$

$$\phi_1(t) = E(t^X) = \exp\big(\lambda(t-1)\big), \quad \varphi_2(t) = E\big(e^{tX}\big) = \exp\big(\lambda\big(e^t - 1\big)\big), \quad \mu_2 = E(X) = \lambda, \quad \sigma_2^2 = \mathrm{Var}\,(X) = \lambda.$$

3. $X =$ **Negative Binomial** (r, p) - the number of trials with probability of success p until r^{th} success. $Y = X - r =$ **Shifted Negative Binomial** (r, p) - the number of failures before r^{th} success. We have

$$P(X = k) = \binom{k-1}{r-1} p^r q^{k-r} \quad \text{for} \quad k = r, r+1, \ldots;$$

$$P(Y = j) = P(X = r + j) = \binom{r+j-1}{r-1} p^r q^j = \binom{r+j-1}{j} p^r q^j \quad \text{for} \quad j = 0, 1, 2, \ldots.$$

Note that by Taylor's formula, for $a \in \mathbb{R}^1$ and $|t| < 1$,

$$g(t) = (1+t)^a = 1 + \sum_{k=1}^{\infty} \frac{g^{(k)}(0)}{k!} \cdot t^k = 1 + \sum_{k=1}^{\infty} \binom{a}{k} t^k, \quad \text{where} \quad \binom{a}{k} = \frac{a(a-1)\cdots(a-k+1)}{k!}.$$

If $a = n$ is a natural number, then this equality is reduced to the binomial formula (for all $t \in \mathbb{R}^1$):

$$g(t) = (1+t)^n = 1 + \sum_{k=1}^{n} \binom{n}{k} t^k, \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

Substituting $t = -q$ and $a = -r$, we get

$$1 = p^r (1-q)^{-r} = p^r \cdot \Big(1 + \frac{r}{1!}\, q + \frac{r(r+1)}{2!}\, q^2 + \cdots\Big) = \sum_{j=0}^{\infty} P(Y=j).$$

Note that the above distributions are well defined for all $r \in \mathbb{R}^1$. correspondingly,

$$\phi(t) = E(t^Y) = \sum_{j=0}^{\infty} t^j P(Y=j) = \sum_{j=0}^{\infty} \binom{r+j-1}{j} p^r (tq)^j = \Big(\frac{p}{1-qt}\Big)^r;$$

$$\varphi_Y(t) = E\big(e^{tY}\big) = \Big(\frac{p}{1-q\,e^t}\Big)^r, \quad \varphi_X(t) = E\big(e^{tX}\big) = e^{tr} E\big(e^{tY}\big) = \Big(\frac{p}{e^{-t}-q}\Big)^r;$$

$$\mu_Y = E(Y) = \frac{rq}{p}, \quad \mu_X = E(X) = \mu_Y + r = \frac{r}{p}, \quad \sigma_X^2 = \mathrm{Var}\,(X) = \mathrm{Var}\,(Y) = \frac{rq}{p^2}.$$

3a. Geometric (p) = **Negative Binomial** $(1, p)$.

4. X = **Gamma** (α, λ), where $\alpha > 0$, $\lambda > 0$, if it has density

$$f_4(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}\, x^{\alpha-1} e^{-\lambda x} \quad \text{for} \quad x > 0, \quad \text{and} \quad f_4(x) = 0 \quad \text{otherwise.}$$

Here $\Gamma(\alpha)$ denotes the Gamma function:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx,$$

which satisfies the properties

$$\Gamma(\alpha+1) = \alpha\,\Gamma(\alpha), \quad \Gamma(n+1) = n!, \quad \Gamma(1/2) = \sqrt{\pi}.$$

We have

$$\varphi_4(t) = E\big(e^{tX}\big) = \Big(\frac{\lambda}{\lambda - t}\Big)^\alpha, \quad \mu_4 = E(X) = \frac{\alpha}{\lambda}, \quad \sigma_4^2 = \mathrm{Var}\,(X) = \frac{\alpha}{\lambda^2}.$$

4a. Exponential (λ) = **Gamma** $(1, \lambda)$.

5. X = **Normal** (μ, σ^2) is related to Y = **Standard Normal** = **Normal** $(0,1)$ by the formula $X = \mu + \sigma \cdot Y$. The corresponding densities

$$f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-y^2/2}, \qquad f_X(x) = \frac{1}{\sigma}\, f_Y\Big(\frac{x-\mu}{\sigma}\Big) = \frac{1}{\sigma\sqrt{2\pi}} \exp\Big[-\frac{(x-\mu)^2}{\sigma^2}\Big].$$

We have

$$\varphi_Y(t) = E\big(e^{tY}\big) = e^{t^2/2}, \qquad \varphi_X(t) = E\big(e^{tX}\big) = \exp\Big(t\mu + \frac{t^2\sigma^2}{2}\Big);$$

$$\mu_Y = E(Y) = 0, \quad \sigma_Y^2 = \mathrm{Var}\,(Y) = 1; \qquad \mu_X = E(X) = \mu, \quad \sigma_X^2 = \mathrm{Var}\,(X) = \sigma^2.$$