MATH 4512. Differential Equations with Applications.
Information to Final Exam on Wednesday, May 11, 2016,
FordH 110, 8:00 am – 10:00 am.

You will have 2 hours to work on 6 problems, some of which are subdivided into subproblems. NO BOOKS, NOTES, and ELECTRONIC DEVICES. You are supposed to know formulas and methods which appeared in homework assignments and Midterm Exams, 4 problems will be based on this material. The remaining 2 problems belong to Sections 7.5–7.7 and 7.9. Some typical problems are as follows:

Sec. 7.5: # 12, 14 (p. 405);
Sec. 7.6: # 9, 10 (p. 417). In the textbook, $t \to \infty$ means $t \to +\infty$;
Sec. 7.7: # 3, 6, 11 (p. 427–428);
Sec. 7.9: # 4, 5, 6, 8 (p. 447).

Some of problems may require analysis, not just calculation. Here are a few examples of such problems.

1. Let $p_1(t)$ and $p_2(t)$ be continuous functions such that $0 < p_1(t) < p_2(t)$ for all real t, and let $y_1(t)$ and $y_2(t)$ satisfy the equations

$$y_1'' + p_1 y_1 = 0, \qquad y_2'' + p_2 y_2 = 0.$$

Suppose that $y_1(t_1) = y_1(t_2) = 0$ at some points $t_1 < t_2$. Show that there is a point t_0 in $[t_1, t_2]$ at which $y_2(t_0) = 0$.

2. Second order equations $y'' = f(y, y')$, where $y = y(t)$ and t does not appear explicitly in the equation, is reduced to first order equations by substitution

$$y'(t) = z(y), \quad y''(t) = \frac{dz(y)}{dt} = z'(y) \cdot y'(t) = zz'.$$

Use this substitution to find the general solution of the equation $(y-1)y'' = 2(y')^2$.

In the remaining problems below, A is a real $n \times n$ matrix.

3. Denote

$$||A|| = \max_{||x|| \le 1} ||Ax||, \quad \text{where} \quad ||x|| \quad \text{is the length of } n\text{-dimensional vector} \quad x$$

(see p. 372). Show that $||A^k|| \le ||A||^k$ for all $k = 1, 2, \ldots$, and $||e^A|| \le e^{||A||}$. By definition, $e^A = I + \sum\limits_{n=1}^{\infty} \frac{A^n}{n!}$.

4. Let A be symmetric, i.e. $A^T = A$, and let v_1 and v_2 be eigenvectors corresponding to two real **distinct** eigenvalues λ_1 and λ_2 ($\lambda_1 \ne \lambda_2$). Show that v_1 and v_2 are orthogonal, i.e. $(v_1, v_2) = 0$.

5. Let $v_1, v_2, \ldots, v_n$ be eigenvectors of a square matrix A corresponding to **distinct** eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Show that they are linearly independent.

6. Let $r_1, r_2, \ldots, r_n$ be distinct roots of the **characteristic equation** $\chi(r) = \det(rI - A) = 0$. Show that $\chi(A) = 0$. (This is a particular case of a general Caley-Hamilton theorem which is applied to every square matrix A).

Hint. If $Ax = rx$, then $\chi(A)x = \chi(r)x$.