MATH 4512. Differential Equations with Applications.
Final Exam. May 11, 2016. Problems and Solutions

Problem 1. Let $p(t)$ be a continuous function such that $0 < p(t) < 1$ for all real t, and let $y(t)$ be a solution of the equation

$$y'' + p(t)y = 0.$$

Suppose that $y(t_1) = y(t_2) = 0$ at some points $t_1 < t_2$. Show that $t_2 - t_1 \geq \pi$, unless $y(t) \equiv 0$.

Proof. Suppose otherwise, i.e. $y(t_1) = y(t_2) = 0$ at some points $t_1 < t_2$ with $0 < t_2 - t_1 < \pi$. In a simple case $y \equiv 0$ on (t_1, t_2), we also have $y' \equiv 0$ on (t_1, t_2), and by uniqueness of solutions, $y(t) = 0$ for all real t.

In the remaining case, when $y(t)$ is not identically 0, we can assume that $y(t) > 0$ at some point $t \in (t_1, t_2)$, because otherwise we just replace y by $-y$. Pick a point a such that $[t_1, t_2]$ lies strictly inside of $(a, a + \pi)$, so that the function $\sin(t - a)$ is strictly positive on $[t_1, t_2]$. Then the function

$$f(t) = \frac{y(t)}{\sin(t-a)} \quad \text{satisfies} \quad f(t_1) = f(t_2) = 0, \quad \text{and} \quad 0 < M = \max_{[t_1,t_2]} f = f(t_0)$$

at some point $t_0 \in (t_1, t_2)$. Further, the function

$$g(t) = y(t) - M\sin(t-a) \leq 0 \quad \text{on} \quad [t_1, t_2], \quad \text{and} \quad g(t_0) = 0.$$

Geometrically, this simply means that we choose an arc of the graph of $M\sin(x - a)$, which touches the graph of $y(t)$ from above at a point $t_0 \in (t_1, t_2)$. Note that

$$y(t_0) = g(t_0) + M\sin(t_0 - a) = M\sin(t_0 - a) > 0.$$

Since $g(t)$ attains its maximum at an interior point t_0, we get

$$0 \geq g''(t_0) = y''(t_0) + M\sin(t_0 - a) = (1 - p)y(t_0) > 0.$$

This contradiction proves that $t_2 - t_1 \geq \pi$. □

Problem 2. Find the general solution of the equation

$$(y-1)y'' = 2(y')^2, \quad \text{where} \quad y = y(t).$$

Solution. Using substitution $y'(t) = z(y)$, we get

$$y''(t) = \frac{dz(y)}{dt} = \frac{dz}{dy} \cdot \frac{dy}{dt} = z'z. \qquad (y-1)z'z = 2z^2.$$

A simple case (a) $z \equiv 0$ corresponds to solutions $y = C = \text{const}$. In the remaining case (b) $z \not\equiv 0$, we can cancel both sides by z, which implies

$$(y-1) \cdot \frac{dz}{dy} = 2z, \qquad \frac{dz}{z} = \frac{2dy}{y-1}, \qquad \ln|z| = 2\ln|y-1| + C,$$

$$\frac{dy}{dt} = z = C_1(y-1)^2, \qquad (y-1)^{-2} = C_1 dt, \qquad (y-1)^{-1} = C_1 t + C_2.$$

In the last equality, we've changed the sign of C_1. Finally, we get $y = 1+(C_1t+C_2)^{-1}$. This is "almost" the final answer, because the case (a) is contained here for $C_1 = 0$, with an exception of $y = 1$, which formally corresponds to $C_2 = \infty$.

Problem 3. Find the general solution of the differential equation

$$y'' + 4y' + 5y = e^{-2t} \sin t.$$

Solution. The characteristic equation $\chi(r) = r^2 + 4r + 5 = 0 = (r + 2)^2 + 1 = 0$ has zeros $r_{1,2} = -2 \pm i$. Note that $e^{r_1 x} = e^{-2x}(\cos x + i \sin x)$, hence $e^{-2x} \sin x = \operatorname{Im}\left(e^{r_1 x}\right)$. Therefore, one can find a particular solution of the given equation in the form $Y = \operatorname{Im} Z$, where Z is a particular solution of

$$Lz = (D^2 + 4D + 5)z = z'' + 4z' + 5z = e^{r_1 x}.$$

Since $r_1 = -2 + i$ is a root of multiplicity 1, one can find Z in the form $Z = Axe^{r_1 x}$. Using the general formula

$$\chi(D)\big(e^{rx} f\big) = e^{rx} \chi(D + r)f \quad \text{with} \quad \chi(D) = (D - r_1)(D - r_2),$$

we get

$$\begin{aligned} LZ &= \chi(D)\big(Axe^{r_1 x}\big) = e^{r_1 x} \chi(D + r_1)(Ax) = e^{r_1 x} D(D + r_1 - r_2)(Ax) = e^{r_1 x} \cdot 2Ai, \\ A &= \frac{1}{2i} = -\frac{i}{2}, \qquad Z = -\frac{i}{2} \cdot xe^{r_1 x} = \frac{1}{2} \cdot xe^{-2x}(\sin x - i \cos x), \qquad Y = \operatorname{Im} Z = -\frac{1}{2} \cdot xe^{-2x} \cos x. \end{aligned}$$

Finally, general solution

$$y(x) = e^{-2x}(C_1 \cos x + C_2 \sin x) - \frac{1}{2} \cdot xe^{-2x} \cos x.$$

Problem 4. Use Laplace transforms to solve the equation

$$y'' + y = \sin t + (\sin t) * y(t), \quad \text{where} \quad (\sin t) * y(t) = \int_0^t \sin(t - \tau)\, y(\tau)\; d\tau,$$

with the initial conditions $y(0) = 0$, $y'(0) = 1$.

Solution. Denote $Y(s) = \mathcal{L}\{y\}$ – the Laplace transform of $y(t)$. Using the equalities

$$\mathcal{L}\{f * g\} = \mathcal{L}\{f\} \cdot \mathcal{L}\{g\}, \qquad \mathcal{L}\{\sin t\} = \frac{1}{s^2 + 1}, \qquad \mathcal{L}\{y''\} = s\mathcal{L}\{y\} - sy(0) - y'(0),$$

we derive

$$(s^2 + 1)Y(s) - 1 = \frac{1}{s^2 + 1} + \frac{1}{s^2 + 1} \cdot Y(s).$$

This equality can be simplified as follows:

$$\big[(s^2 + 1)^2 - 1\big]\, Y(s) = s^2 + 2, \qquad (s^4 + 2s^2)Y(s) = s^2 + 2, \qquad Y(s) = s^{-2},$$

which corresponds to $y(t) = t$.

Problem 5. Find the general solution of the system

$$\frac{dx_1}{dt} = x_2 + \tan^2 t - 1, \qquad \frac{dx_2}{dt} = -x_1 + \tan t.$$

Solution. Differentiate the second equality and substitute x_1' from the first equality:

$$x_2'' = (-x_1 + \tan t)' = -x_1' + \frac{1}{\cos^2 t} = -x_2 - \tan^2 t + 1 + \frac{1}{\cos^2 t} = -x_2 + 2.$$

The general solution of $x_2'' + x_2 = 2$ is $x_2 = C_1 \cos t + C_2 \sin t + 2$.
Finally, $x_1 = -x_2' + \tan t = C_1 \sin t - C_2 \cos t + \tan t$.

Problem 6. If $A = \begin{pmatrix} 5 & 8 \\ 2 & 5 \end{pmatrix}$, find
(a) the inverse matrix A^{-1};
(b) the eigenvalues and eigenvectors of A;
(c) the matrix function e^{tA}.

Solution. (a). We have

$$\det A = 25 - 16 = 9, \quad \text{and} \quad A^{-1} = \frac{1}{\det A} \begin{pmatrix} 5 & -8 \\ -2 & 5 \end{pmatrix} = \begin{pmatrix} 5/9 & -8/9 \\ -2/9 & 5/9 \end{pmatrix}.$$

(b). The eigenvalues of A are roots of the characteristic equation

$$\chi(\lambda) = \det(\lambda I - A) = \det \begin{pmatrix} \lambda - 5 & -8 \\ -2 & \lambda - 5 \end{pmatrix} = \lambda^2 - 10\lambda + 9 = (\lambda - 1)(\lambda - 9),$$

i.e. $\lambda_1 = 1$, $\lambda_2 = 9$. The corresponding eigenvectors are nonzero solutions of systems $(\lambda I - A)v = 0$:

$$\lambda_1 = 1 \quad \text{corresponds to} \quad v_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \qquad \lambda_2 = 9 \quad \text{corresponds to} \quad v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

(c). A fundamental matrix Ψ with columns $e^{\lambda_1 t} v_1$ and $e^{\lambda_2 t} v_2$ satisfies the matrix equation $\Psi' = A\Psi$. The exponential matrix

$$\begin{aligned} e^{tA} = \Psi(t) \cdot \Psi(0)^{-1} &= \begin{pmatrix} 2e^t & 2e^{9t} \\ -e^t & e^{9t} \end{pmatrix} \cdot \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{4} \cdot \begin{pmatrix} 2e^t & 2e^{9t} \\ -e^t & e^{9t} \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix} \\ &= \frac{1}{4} \cdot \left[e^t \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} + e^{9t} \begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix} \right]. \end{aligned}$$