# Exponential Matrix and Stability of Systems.

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## 1 Exponential Matrix.

**Definition 1.1.** For $n \times n-$matrix $A$, the **exponential matrix function**

$$e^{tA} = I + \frac{t}{1!}A + \frac{t^2}{2!}A^2 + \cdots = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k. \tag{1.1}$$

**Lemma 1.2.** $X = X(t) = e^{tA}$ is the unique solution of the Cauchy problem

$$X' = \frac{dX}{dt} = AX, \qquad X(0) = I. \tag{1.2}$$

**Proof.** It is easy to see that

$$\frac{d}{dt}e^{tA} = A + \frac{t}{1!}A^2 + \cdots = Ae^{tA},$$

so that $X = e^{tA}$ is a solution of (1.2). Moreover, we can treat $n \times n-$matrix function $X$ as a vector functions with values in $\mathbb{R}^{n^2}$ (or $\mathbb{C}^{n^2}$). we only need to rewrite the matrix equation $X' = AX$ in the vector form $X' = BX$ with a $n^2 \times n^2-$matrix $B$. Then the uniqueness for the Cauchy problem in the vector form implies the uniqueness for the problem (1.2). $\square$

**Definition 1.3.** If $AV = \lambda V$, for some vector $V \neq 0$, then $\lambda$ is an **eigenvalue** of $A$, and $V$ is an **eigenvector** corresponding to $\lambda$.

We have $AV = \lambda V \iff (A - \lambda I)V = 0$. The last system has nontrivial solutions $V \neq 0 \iff |A - \lambda I| = 0$. We introduce the **characteristic** polynomial of $A$ by the formula $p(\lambda) = |\lambda I - A|$. Now we can conclude:

**(i)** The eigenvalues of $A$ are roots of the **characteristic** equation

$$p(\lambda) = |\lambda I - A| = 0. \tag{1.3}$$

**(ii)** For each eigenvalue $\lambda$, the corresponding eigenvectors $V$ are nontrivial solutions of the system

$$(A - \lambda I)V = 0. \tag{1.4}$$

**Lemma 1.4.** Let $A$ be a constant $n \times n$ matrix, and let $AV = \lambda V$ for some vector $V$. Then the matrix function $U = U(t) = e^{\lambda t}V$ satisfies $U' = AU$.

**Proof.** $U'(t) = \left(e^{\lambda t}\right)' V = e^{\lambda t}\lambda V = e^{\lambda t} AV = AU(t)$. $\square$

**Lemma 1.5.** For distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$ of the matrix $A$, the corresponding eigenvectors $V_1, V_2, \cdots, V_m$ are linearly independent.

**Proof.** For $m = 1$, this is trivial:

$$V_1 \neq 0, \quad c_1 V_1 = 0 \quad \Rightarrow \quad c_1 = 0.$$

Now suppose this statement is true for some $m = k$, i.e. $V_1, V_2, \cdots, V_k$ are linearly independent. We will prove that it remains true for $m = k + 1$, i.e. the equality

$$c_1 V_1 + \cdots + c_k V_k + c_{k+1} V_{k+1} = 0 \tag{1.5}$$

holds only in case $c_1 = \cdots = c_k = c_{k+1} = 0$. Multiplying (1.5) by the matrix $A$ and using the equalities $AV_j = \lambda_j V_j$, we get

$$c_1 \lambda_1 V_1 + \cdots + c_k \lambda_k V_k + c_{k+1} \lambda_{k+1} V_{k+1} = 0.$$

Now subtract (1.5) multiplied by $\lambda_{k+1}$. This gives us

$$c_1 \left(\lambda_1 - \lambda_{k+1}\right) V_1 + \cdots + c_k \left(\lambda_k - \lambda_{k+1}\right) V_k = 0.$$

By our assumption, $V_1, V_2, \cdots, V_k$ are linearly independent. Therefore,

$$c_1 \left(\lambda_1 - \lambda_{k+1}\right) = \cdots = c_k \left(\lambda_k - \lambda_{k+1}\right) = 0.$$

Since all the eigenvalues are distinct, this implies $c_1 = \cdots = c_k = 0$. Now from (1.5) it follows $c_{k+1} = 0$. This proves our statement for $m = k + 1$, and by induction, it is true for arbitrary $m$. $\square$

Another way of computation $e^{tA}$ is based on the following famous result.

**Theorem 1.6 ( Cayley-Hamilton).** Let $A = (a_{ij})$ be a $n \times n$ matrix, and let $p(\lambda) = |\lambda I - A|$. Then $p(A) = 0$.

**Proof.** Denote

$$p(\lambda) = |\lambda I - A| = \sum_{j=0}^{n} p_j \lambda^j, \qquad (p_n = 1).$$

Similarly to the geometric series for large $\lambda > 0$, we have a convergent series

$$(\lambda I - A)^{-1} = \frac{1}{\lambda} \cdot \left(I - \frac{1}{\lambda} \cdot A\right)^{-1} = \frac{1}{\lambda} \cdot \sum_{k=0}^{\infty} \lambda^{-k} A^k.$$

On the other hand, the elements of this function are co-factors of $\lambda I - A$ divided by $p(\lambda) = |\lambda I - A|$. Therefore, the product

$$p(\lambda) \cdot (\lambda I - A)^{-1} = \frac{1}{\lambda} \cdot \sum_{j,k} p_j \lambda^{j-k} A_k$$

is a polynomial of $\lambda$ with matrix coefficient, so that the coefficients of all negative powers of $\lambda$ must be 0. In particular, the coefficient of $\lambda^{-1}$, which corresponds to a part of the above sum for $j = k$, is

$$0 = \sum_j p_j A^j = p(A).$$

$\square$

Let $\lambda_1, \lambda_2, \cdots, \lambda_s$ denote all the distinct eigenvalues of a matrix $A$ (real and complex). Then we can write the characteristic polynomial of $A$ in the form

$$p(\lambda) = |\lambda I - A| = \prod_{k=1}^{s} (\lambda - \lambda_k)^{m_k}, \tag{1.6}$$

where $m_k$ is the multiplicity of $\lambda_k$. We have

$$\frac{1}{p(\lambda)} = \sum_{k=1}^{s} \frac{a_k(\lambda)}{(\lambda - \lambda_k)^{m_k}}, \tag{1.7}$$

where $a_k(\lambda)$ is a polynomial of degree $\leq m_k - 1$ for each $k$. Therefore,

$$1 = \sum_{k=1}^{s} a_k(\lambda) p_k(\lambda), \qquad I = \sum_{k=1}^{s} a_k(A) p_k(A), \tag{1.8}$$

where

$$p_k(\lambda) = \frac{p(\lambda)}{(\lambda - \lambda_k)^{m_k}} = \prod_{j \neq k} (\lambda - \lambda_j)^{m_j}. \tag{1.9}$$

**Theorem 1.6.** Let $\lambda_1, \lambda_2, \cdots, \lambda_s$ be all the distinct eigenvalues of a matrix $A$ with the characteristic polynomial $p(\lambda)$ in (1.6). Then

$$e^{tA} = \sum_{k=1}^{s} e^{\lambda_k t} a_k(A) p_k(A) \sum_{j=0}^{m_k - 1} \frac{t^j}{j!} (A - \lambda_k I)^j, \tag{1.10}$$

where $a_k$ are polynomials of degree $\leq m_k - 1$, which are defined from the decomposition (1.7), and $p_k$ are polynomials in (1.9).

First we prove a lemma, which allows us to use the equalities

$$e^{tA} = e^{t\lambda I} e^{t(A - \lambda A)} = e^{\lambda t} e^{t(A - \lambda A)}. \tag{1.11}$$

**Lemma 1.7.** Let $A$ and $B$ be $n \times n$ matrices satisfying $AB = BA$. Then

$$e^{tA} e^{tB} \equiv e^{t(A+B)}.$$

**Proof of Lemma.** We have

$$AB = BA \implies A^k B = BA^k \implies e^{tA} B = B e^{tA}.$$

Therefore, the matrix function $X(t) = e^{tA} e^{tB}$ satisfies

$$X' = (e^{tA})' e^{tB} + e^{tA} (e^{tB})' = A e^{tA} e^{tB} + B e^{tA} e^{tB} = (A + B) X, \qquad X(0) = I.$$

By Lemma 1.2, we have $X(t) \equiv e^{t(A+B)}$. $\square$

**Proof of Theorem 1.6.** Using the equalities (1.8) and (1.11) we get

$$e^{tA} = \sum_{k=1}^{s} a_k(A)p_k(A)e^{tA} = \sum_{k=1}^{s} e^{\lambda_k t} a_k(A)p_k(A)e^{t(A-\lambda_k I)}. \tag{1.12}$$

Since $p_k(A)(A-\lambda_k I)^{m_k} = p(A) = 0$,

$$p_k(A)e^{t(A-\lambda_k I)} = p_k(A)\sum_{j=0}^{\infty}\frac{t^j}{j!}(A-\lambda_k I)^j = p_k(A)\sum_{j=0}^{m_k-1}\frac{t^j}{j!}(A-\lambda_k I)^j.$$

Together with (1.12), this gives us the equality (1.10). $\square$

**Example 1.8.** Consider the matrix $A = \begin{pmatrix} -1 & 1 & -2 \\ 0 & -1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$. We have

$$p(\lambda) = |\lambda I - A| = (\lambda+1)^2(\lambda-1) \implies$$

$$\lambda_1 = -1, \quad m_1 = 2, \quad \lambda_2 = 1, \quad m_2 = 1, \quad p_1(\lambda) = \lambda - 1, \quad p_2(\lambda) = (\lambda+1)^2;$$

$$\frac{1}{p(\lambda)} = \frac{1}{(\lambda+1)^2(\lambda-1)} = \frac{1}{4}\left(\frac{1}{\lambda-1} - \frac{\lambda+3}{(\lambda+1)^2}\right) \implies a_1(\lambda) = -\frac{1}{4}(\lambda+3), \quad a_2(\lambda) = \frac{1}{4}.$$

By (1.10), we have

$$e^{tA} = -\frac{1}{4}e^{-t}(A+3I)(A-I)\big[I + t(A+I)\big] + \frac{1}{4}e^{t}(A+I)^2.$$

Finally,

$$-\frac{1}{4}(A+3I)(A-I) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \frac{1}{4}(A+I)^2 = \frac{1}{4}\begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 0 & 2 \end{pmatrix}^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\implies \quad e^{tA} = e^{-t}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} + te^{-t}\begin{pmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + e^{t}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Example 1.9.** Consider the matrix $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. We have

$$p(\lambda) = \lambda^2 + 1, \quad \lambda_1 = i, \quad p_1(\lambda) = \lambda + i; \quad \lambda_2 = -i, \quad p_2(\lambda) = \lambda - i, \quad m_1 = m_2 = 1;$$

$$\frac{1}{\lambda^2+1} = \frac{a_1}{\lambda-1} + \frac{a_2}{\lambda+1}, \quad \text{where} \quad a_1 = \lim_{\lambda\to i}\frac{\lambda-i}{\lambda^2+1} = \frac{1}{2i}, \quad a_2 = -\frac{1}{2i};$$

$$e^{tA} = \sum e^{\lambda_k t}a_k p_k(A) = e^{it}\cdot\frac{1}{2i}\cdot(A+iI) + e^{-it}\cdot\frac{-1}{2i}\cdot(A-iI) = \frac{e^{it}-e^{-it}}{2i}\cdot A + \frac{e^{it}+e^{-it}}{2}\cdot I$$

$$= \cos t\cdot I + \sin t\cdot A = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$