Appendix B.
Exponential and Logarithmic Functions

For fixed $b > 1$, the function b^x was defined in Exercise 6 on p.22 in the textbook "Principles of Mathematical Analysis" by W. Rudin. It satisfies $b^x > 0$, and

(E1). $b^{x+y} = b^x b^y$ for real x, y.

In particular, $b^0 = 1$, which implies

$$1 = b^0 = b^{y+(-y)} = b^y b^{-y} \quad \Longleftrightarrow \quad b^{-y} = (b^y)^{-1} \quad \Longleftrightarrow$$

(E2). $b^{x-y} = b^x b^{-y} = b^x / b^y$ for real x, y.

(E3). $(b^x)^y = b^{xy}$ for real x, y.

We divide the proof of this property into a few steps.

Step 1. $y = n$ is natural. Then by iterating of (E1),

$$(b^x)^n = \underbrace{b^x b^x \cdots b^x}_{n\,\text{times}} = b^{xn}.$$

Step 2. $y = -n$, where n is natural. Since $(b^x)^n (b^x)^{-n} = 1$, we get

$$(b^x)^{-n} = \big((b^x)^n\big)^{-1} = (b^{nx})^{-1} = b^{-nx}.$$

Together with the obvious case $y = 0$, the cases 1 and 2 cover all integers y.

Step 3. $y = 1/n$, where n is natural. We have

$$(e^{x/n})^n = b^{nx/n} = b^x \quad \Longleftrightarrow \quad (b^x)^{1/n} = b^{x/n}.$$

Step 4. $y = m/n$ – a rational number. Here m is integer and n is natural. Then

$$(b^x)^y = (b^x)^{m/n} = \big((b^x)^{1/n}\big)^m = (b^{x/n})^m = e^{mx/n} = e^{xy}.$$

Step 5. $x > 0$ and y is real. Then $b_1 := b^x > 1$, and similarly to Ex. 6(c,d) on p.22,

$$(b^x)^y = b_1^y = \sup_{r \le y} b_1^r = \sup_{r \le y} b^{rx} = \sup_{r_1 \le xy} b_1^r = b^{xy}.$$

Here the sup is taken over rational numbers r or r_1.

The assumption $b > 1$ was needed in order to have an increasing function b^x, which is defined as the sup of b^r over rational numbers $r \le x$. If $0 < b < 1$, then $b^{-1} > 1$, and we can define

$$b^x := (b^{-1})^{-x}.$$

For completeness, we also set $1^x \equiv 1$. Then the function b^x is defined for all $b > 0$ and real x, and it satisfies (E1)–(E3). For example, if $0 < b < 1$, then the property (E3) can be verified as follows:

$$(b^x)^y = \big((b^{-1})^{-x}\big)^y = (b^{-1})^{-xy} = b^{xy}.$$

Definition (compare with Ex.7 on p.22). Let $b > 0$, $b \neq 1$, and $y > 0$ be fixed. The **logarithm of y to the base b**,

$$x = \log_b y \quad \text{- the unique solution of} \quad b^x = y, \quad \text{i.e.} \quad b^{\log_b y} \equiv y \quad \text{for} \quad y > 0.$$

The **natural logarithm** of y

$$\ln y := \log y := \log_e y, \quad \text{where} \quad e := \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = 2.71828\ldots.$$

The logarithmic function $\log_b y$ can be easily expressed in terms of the function $\ln y$:

(L0). $\log_b y = \ln y / \ln b$. Indeed,

$$b^{\ln y/\ln b} = \left(e^{\ln b}\right)^{\ln y/\ln b} = e^{\ln b \cdot \ln y/\ln b} = e^{\ln y} = y,$$

and (L0) holds true by definition.

The following properties (L1)–(L3) for $\ln y$ correspond to (E1)–(E3) for e^x. They are true for $\log_b y$ as well.

(L1). $\ln(y_1 y_2) = \ln y_1 + \ln y_2$ for $y_1 > 0$, $y_2 > 0$.
This equality follows from

$$e^{\ln y_1 + \ln y_2} = e^{\ln y_1} e^{\ln y_2} = y_1 y_2 = e^{\ln(y_1 y_2)}.$$

(L2). $\ln(y_1/y_2) = \ln y_1 - \ln y_2$ for $y_1 > 0$, $y_2 > 0$.
The proof is quite similar to the previous one.

(L3). $\ln(y^a) = a \ln y$ for $y > 0$ and real a.
Indeed, using (E3), we obtain

$$e^{a \ln y} = (e^{\ln y})^a = y^a = e^{\ln(y^a)},$$

which is equivalent to (L3).

One an also prove (L2) by combining (L1) and (L3) with $a = -1$:

$$\ln(y_1/y_2) = \ln(y_1 \cdot y_2^{-1}) = \ln(y_1) + \ln(y_2^{-1}) = \ln y_1 - \ln y_2.$$

Using the properties (E) and (L), we also get a new property

(E4). $(ab)^x = a^x b^x$ for $a > 0$, $b > 0$ and real x.

It suffices to check that the logarithms of both sides coincide, and this is the case:

$$\ln\left((ab)^x\right) = x \cdot \ln(ab) = x \cdot \ln a + x \cdot \ln b = \ln(a^x) + \ln(b^x) = \ln(a^x b^x).$$

The base e of the natural logarithm satisfies some special properties.

Theorem. The sequences

$$a_n := \left(1 + \frac{1}{n}\right)^n \nearrow e := 2.71828\ldots, \quad b_n := \left(1 + \frac{1}{n}\right)^{n+1} \searrow e \quad \text{as} \quad n \to \infty.$$

Proof. We will use an elementary inequality, which is easily proved by induction:

$$(1 + h)^n \geq 1 + nh \quad \text{for all} \quad h \geq -1 \quad \text{and} \quad n = 1, 2, 3, \ldots.$$

Using this inequality for $n \geq 2$ and $h := -1/n^2$, we get

$$\frac{a_n}{a_{n-1}} = \left(\frac{n+1}{n}\right)^n \left(\frac{n-1}{n}\right)^{n-1} = \left(1 - \frac{1}{n^2}\right)^n \cdot \frac{n}{n-1} \geq \left(1 - \frac{n}{n^2}\right) \cdot \frac{n}{n-1} = 1,$$

i.e. $a_{n-1} \leq a_n$ for all $n \geq 2$. Similarly,

$$\begin{aligned} \frac{b_{n-1}}{b_n} &= \left(\frac{n}{n-1}\right)^n \left(\frac{n}{n+1}\right)^{n+1} = \left(\frac{n^2}{n^2-1}\right)^{n+1} \cdot \frac{n-1}{n} \\ &= \left(1 + \frac{1}{n^2-1}\right)^{n+1} \cdot \frac{n-1}{n} \geq \left(1 + \frac{n+1}{n^2-1}\right) \cdot \frac{n-1}{n} = 1, \end{aligned}$$

i.e. $b_{n-1} \geq b_n$ for all $n \geq 2$.

Note that since n cannot have nontrivial common factors with $n-1$ or $n+1$, we actually have strict inequalities "$>$" instead of "$\geq$" in the above expressions:

$$a = a_1 < a_2 < a_3 < \cdots < a_n < b_n < \cdots < b_3 < b_2 < b_1 = 4.$$

By Theorem 3.19, there exists $\lim a_n$, which we denote by e. We also have

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) a_n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \cdot \lim_{n\to\infty} a_n = 1 \cdot e = e.$$

$\square$

Corollary. We have

$$\frac{1}{n+1} < \ln\left(1 + \frac{1}{n}\right) < \frac{1}{n} \quad \text{for all} \quad n = 1, 2, 3, \ldots.$$

Proof. Since $\ln y$ is an increasing function for $y > 0$, from the previous theorem it follows $\ln a_n < \ln e = 1 < \ln b_n$. Using the property (L3) of $\ln y$, we get

$$n \cdot \ln\left(1 + \frac{1}{n}\right) < 1 < (n+1) \cdot \ln\left(1 + \frac{1}{n}\right),$$

and the desired inequalities follow. $\square$