Appendix C. Integration in the Complex Plane.

The theory of integration of functions $f(z)$ of complex variable $z = x + iy \in \mathbb{C}$ was mainly developed by a French mathematician Augustin-Louis Cauchy (1789–1857). Here we present some of its key points.

If γ is a piecewise differentiable curve parameterized as $z = z(t),\ a \le t \le b$, and $f(z)$ is a continuous function on γ, then the **line integral** of f along γ is defined as

$$\int_\gamma f(z)\,dz := \int_a^b f(z(t))z'(t)\,dt.$$

One can always change parametrization to arbitrary $a < b$, e.g. $a = 0$ and $b = 1$ or 2π. We have

$$\left|\int_\gamma f(z)\,dz\right| \le |\gamma| \cdot \sup_\gamma |f|, \quad \text{where} \quad |\gamma| := \int_a^b |z'(t)|\,dt \quad \text{is the length of} \quad \gamma. \tag{1}$$

Example 1. If $\gamma = \{|z - z_0| = \rho = \text{const} > 0\}$, then we can parameterize it as $z(t) = z_0 + \rho e^{it}$, $0 \le t \le 2\pi$. Then for integers n,

$$\int_\gamma (z - z_0)^n dz = \int_0^{2\pi} (\rho e^{it})^n i\rho\, e^{it} dt = i\rho^{n+1} \int_0^{2\pi} e^{i(n+1)t} dt = \begin{cases} 2\pi i & \text{for} \quad n = -1, \\ 0 & \text{for} \quad n \ne -1. \end{cases}$$

Theorem 1. *Let γ_s, $0 \le s \le 1$, be a family of smooth curves connecting two points z_0 and z_1 in an open set $D \subset \mathbb{C}$, in a sense that γ_s are parameterized as*

$$\gamma_s: \quad z = z(s,t), \quad 0 \le t \le 1, \quad \text{with} \quad z_0 = z(s,0) \quad \text{and} \quad z_1 = z(s,1),$$

and $z(s,t)$ is twice continuously differentiable in $s, t \in [0,1]$, with values in D. Let $f(z)$ be a continuously differentiable function in D. Then

$$I(s) := \int_{\gamma_s} f(z)\,dz = \text{const} \quad \text{for} \quad 0 \le s \le 1.$$

Remark 1. Smoothness of $z(s,t)$ is a technical assumption. It is important only that γ_s move continuously for $0 \le s \le 1$ and do not cross the point of singularity of f at which its derivative f' is not defined.

Proof. We can write

$$\frac{dI(s)}{ds} = \frac{d}{ds} \int_0^1 f(z) \cdot \frac{\partial z}{\partial t} \cdot dt = \int_0^1 \frac{\partial}{\partial s}\Big[f(z) \cdot \frac{\partial z}{\partial t}\Big]\,dt.$$

By the chain rule, the integral functions is equal to

$$f'(z) \cdot \frac{\partial z}{\partial s} \cdot \frac{\partial z}{\partial t} + f(z) \cdot \frac{\partial^2 z}{\partial s\,\partial t}.$$

Since this expression is symmetric, one can interchange s and t, so that

$$\frac{dI(s)}{ds} = \int_0^1 \frac{\partial}{\partial t}\Big[f(z)\cdot\frac{\partial z}{\partial s}\Big]\,dt = f(z)\cdot\frac{\partial z}{\partial s}\Big|_{t=0}^{t=1} = 0,$$

because $z(s,0) = z_0$ and $z(s,1) = z_1$ do not depend on s. Therefore, $I(s) = \text{const}$. □

Theorem 2 (The Cauchy Integral Formula). *If f is continuously differentiable function in D, and a closed contour*

$$C := \big\{z(t),\ a \le t \le b, \quad \text{with} \quad z(0) = z(1)\big\}$$

can be continuously transformed to a single point z_0 inside of C, then

$$f(z_0) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz. \tag{2}$$

Integral in (2) denotes the line integral along the closed contour C with counterclockwise orientation and without self-intersections. We call such piecewise smooth contours **simple contours**.

Proof. By the previous theorem, one can replace C by $\gamma_\varepsilon := \{|z - z_0| = \varepsilon\} \subset D$ with small $\varepsilon > 0$. Since $|\gamma_\varepsilon| = 2\pi\varepsilon$, using Example 1 together with the estimate (1), we get

$$\left|\frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0}\,dz - f(z_0)\right| = \left|\frac{1}{2\pi i}\oint_{\gamma_\varepsilon} \frac{f(z) - f(z_0)}{z - z_0}\,dz\right|$$

$$\le \frac{1}{2\pi}\cdot\sup_{\gamma_\varepsilon}\left|\frac{f(z) - f(z_0)}{z - z_0}\right|\cdot|\gamma_\varepsilon| = \sup_{\gamma_\varepsilon}\big|f(z) - f(z_0)\big| \to 0 \quad \text{as} \quad \varepsilon \searrow 0.$$

Since the left hand side does not depend on ε, we must have the equality (2). □

Remark 2. If a disc $B_r(z_0)$ lies inside of C, then

$$f(z_0 + h) = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z - z_0 - h}\,dz \quad \text{for} \quad |h| < r.$$

Here

$$\frac{1}{z - z_0 - h} = \frac{1}{z - z_0}\cdot\frac{1}{1 - \frac{h}{z - z_0}} = \sum_{n=0}^{\infty}\frac{h^n}{(z - z_0)^{n+1}}.$$

This series converges uniformly for $z \in C$, because $|h| < r \le |z - z_0|$. This implies that a continuously differentiable function $f(z)$ is in fact an **analytic** function, and the corresponding power series has radius of convergence $R \ge r > 0$.

The following theorem is convenient for many applications. It can be considered as a special form of the **Cauchy residue theorem**.

Theorem 3. *Let f and g be analytic functions in a domain D, and let g have finitely many zeros $z_1, z_2, \ldots, z_n$ inside of a simple contour $C \subset D$. Suppose that*

$$g(z_k) = 0 \ne g'(z_k) \quad \text{for} \quad k = 1, 2, \ldots, n.$$

Then

$$\oint\limits_C \frac{f(z)}{g(z)}\,dz = 2\pi i \cdot \sum_{k=1}^{n} \frac{f(z_k)}{g'(z_k)}. \tag{3}$$

Proof. The contour C encircles a subdomain $D_0 \subset D$, so that $C = \partial D_0$ is the boundary of D_0. One can subdivide D_0 into subdomains D_k with piecewise smooth boundaries $C_k := \partial D_k$, such that $z_k \in D_k$ for $k = 1, 2, \ldots, n$. Then the integral along C in (3) will be represented as the sum of integrals along C_k, because the integrals along the common parts of C_k inside of D_0 cancel. In turn, each of integrals along C_k can be reduced to integrals along $\gamma_{k,\varepsilon} := \{|z - z_k| = \varepsilon\} \subset D_k$ with small $\varepsilon > 0$. Therefore, it suffices to show that

$$\oint\limits_{\gamma_{k,\varepsilon}} \frac{f(z)}{g(z)} dz = 2\pi i \cdot \frac{f(z_k}{g'(z_k)} \quad \text{for} \quad k = 1, 2, \ldots, n. \tag{4}$$

According to Remark 2, $g(z)$ is an analytic function in a neighborhood of z_k, and since $g(z_k) = 0$, it can be represented as

$$g(z) = \sum_1^\infty c_n \cdot (z - z_k)^n = (z - z_k) \cdot \sum_1^\infty c_n \cdot (z - z_k)^{n-1} =: (z - z_k) \cdot g_1(z_k).$$

Here $g_1(z)$ is also an analytic functions in a neighborhood of z_k with $g_1(z_k) = c_1 = g'(z_k) \neq 0$. One can choose $\varepsilon > 0$ so small that $g_1(z) \neq 0$ for $|z - z_k| \leq \varepsilon$. By Theorem 2, we how get (4):

$$\oint\limits_{\gamma_{k,\varepsilon}} \frac{f(z)}{g(z)} dz = \oint\limits_{\gamma_{k,\varepsilon}} \frac{f(z)/g_1(z)}{z - z_k} dz = 2\pi i \cdot \frac{f(z_k)}{g_1(z_k)} = 2\pi i \cdot \frac{f(z_k)}{g'(z_k)}.$$

Theorem is proved. □

Example 2. We show that

$$I_n := \int\limits_{-\infty}^{\infty} \frac{dx}{1 + x^{2n}} = \frac{\pi}{n \cdot \sin\left(\frac{\pi}{2n}\right)} \quad \text{for} \quad n = 1, 2, \ldots. \tag{5}$$

First note that the function $g(z) := 1 + z^{2n}$ has $2n$ zeros

$$z_k := z_1 \cdot q^{k-1}, \quad \text{where} \quad z_1 := e^{\pi i/2n}, \quad q := e^{\pi/n}, \quad k = 1, 2, \ldots, 2n.$$

For $R > 1$, n of these zeros, $z_1, z_2, \ldots, z_n$, lie inside of the half-disc $D_R^+ := \{x \in \mathbb{C} : \ |z| \leq R, \ \mathrm{Im}\, z \geq 0\}$. By Theorem 3,

$$\oint\limits_{\partial B_R^+} \frac{dz}{1 + z^{2n}} = 2\pi i \cdot \sum_{k=1}^{n} \frac{1}{2n \cdot z_k^{2n-1}}, \quad R > 1.$$

The integral in the left hand side can be represented as the sum of the integral along $[-R, R]$, which converges to I_n as $R \to +\infty$, and the integral along $\gamma_R := \{|z| = R, \ \mathrm{Im}\, z \geq 0\}$, which is easily estimated by (1):

$$\left| \int\limits_{\gamma_R} \frac{dz}{1 + z^{2n}} \right| \leq |\gamma_R| \cdot \sup_{\gamma_R} \frac{1}{|1 + z^{2n}|} \leq \frac{\pi R}{R^{2n-1}} \to 0 \quad \text{as} \quad R \to +\infty.$$

Since $z_k^{2n} = -1$, we get (5):

$$\begin{aligned} I_n &= \oint_{\partial B_R^+} \frac{dz}{1+z^{2n}} = \frac{-\pi i}{n} \cdot \sum_{k=1}^{n} z_k = \frac{-\pi i\, z_1}{n} \cdot \sum_{k=1}^{n} q^{k-1} \\ &= \frac{-\pi i\, z_1}{n} \cdot \frac{1-q^n}{1-q} = \frac{-2\pi i\, z_1}{n(1-q)} = \frac{-2\pi i\, z_1}{n(1-z_1^2)} = \frac{2\pi i}{n(z_1 - z_1^{-1})} = \frac{\pi}{n \cdot \sin\left(\frac{\pi}{2n}\right)}. \end{aligned}$$

Remark 3. The equality (5) can also be easily derived by substitution $t = x^{2n}$ from a more general equality with $\alpha = 1/2n$:

$$\Gamma(\alpha)\Gamma(1-\alpha) = \int_0^\infty \frac{t^{\alpha-1}dt}{1+t} = \frac{\pi}{\sin(\pi\alpha)} \quad \text{for} \quad 0 < \alpha < 1.$$

Example 4. As another application of the Cauchy Integral Formula (2), we derive an explicit representation of the **generating function** for the **Legendre polynomials**

$$L_n(x) := \frac{1}{2^n \cdot n!} \left[(x^2-1)^n\right]^{(n)}.$$

We claim that

$$\varphi(t) := \sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{\sqrt{1-2tx+t^2}} \quad \text{for} \quad |x| \le 1 \quad \text{and small} \quad |t|. \tag{6}$$

For fixed x, let C be a circle centered at x. Using formula (2) with $f(z) := (z^2-1)^n$ and $z_0 := x$, we obtain

$$\begin{aligned} (x^2-1)^n &= \frac{1}{2\pi i} \oint_C \frac{(z^2-1)^n}{z-x}\, dz, \\ L_n(x) &= \frac{1}{2\pi i \cdot 2^n \cdot n!} \oint_C \frac{\partial^n}{\partial x^n} \left[\frac{(z^2-1)^n}{z-x}\right] dz = \frac{1}{2\pi i \cdot 2^n} \oint_C \frac{(z^2-1)^n}{(z-x)^{n+1}} dz, \\ \varphi(t) &= \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \left[\frac{t(z^2-1)}{2(z-x)}\right]^n \frac{dz}{z-x} = \frac{1}{2\pi i} \oint_C \left[1 - \frac{t(z^2-1)}{2(z-x)}\right]^{-1} \frac{dz}{z-x} \\ &= \frac{1}{2\pi i} \oint_C \frac{dz}{g(z)}, \quad \text{where} \quad g(z) := z - x - \frac{t}{2} \cdot (z^2-1). \end{aligned}$$

For small $t \neq 0$, the quadratic polynomial $g(z)$ has zeros

$$z_0 := \frac{1-\sqrt{1-2tx+t^2}}{t} \quad \text{and} \quad z_1 := \frac{1+\sqrt{1-2tx+t^2}}{t}.$$

Since $1 - \sqrt{1-\alpha} \sim \alpha/2$ for small $\alpha := 2tx + t^2$, only z_0 lies close to x, i.e. inside C. By virtue of (2), the desired equality (6) follows:

$$\varphi(t) = \frac{1}{g'(z_0)} = \frac{1}{1-tz_0} = \frac{1}{\sqrt{1-2tx+t^2}}.$$