Appendix A. Ordered sets

This note is supplementary to the book:
[1] *G. F. Folland*, "Real analysis. Modern Techniques and Their Applications", 2nd Edition, ©1999 by John Wiley & Sons, Inc.

In our exposition, we partially follow Chapter I, §6 in the book:
[2] *A. G. Kurosh*, "Lectures on General Algebra", ©1963 by Chelsea Publishing Company.

Here we derive the Well Ordering Principle and the Hausdorff Maximal Principle from
The Axiom of Choice. For an arbitrary non-empty set X, there exists a function

$$f : 2^X \setminus \emptyset \mapsto X \quad \text{such that} \quad f(A) \in A \quad \text{for every non-empty set} \quad A \subseteq X. \tag{1}$$

Throughout this note, we assume that X is a non-empty set.

A set X is **partially ordered** by a relation "$x \le y$" for **some** pairs $(x, y) \in X \times X$ if
(i) $x \le y,\ y \le z \quad \Longrightarrow \quad x \le z$;
(ii) $x \le y,\ y \le x \quad \Longrightarrow \quad x = y$;
(iii) $x \le x$ for every $x \in X$.
We write $x < y$ iff $x \le y,\ x \ne y$.

A partially ordered set $(X, \le)$ is **linearly (totally) ordered**, or $(X, \le)$ is a **chain**, if for **every** pair $(x, y) \in X \times X$ we have either $x \le y$ or $y \le x$.

A linearly ordered set $(X, \le)$ is **well ordered** if every non-empty set $A \subseteq X$ contains its **minimal element** $\min A \in A$, i.e. $\min A \le x$ for every $x \in A$.

Further, define a **segment** of a well ordered set $(X, \le)$ to be a subset $S \subseteq X$ such that for every $a \in S$, we also have $[m, a) := \{x \in X :\ x < a\} \subset S$. In addition, S is a **proper segment** of $(X, \le)$ if $X \setminus S$ is non-empty.

Theorem 1 (**Zermelo's Well Ordering Principle**). *Every non-empty set X can be well ordered.*

Proof. Let f be a function satisfying (1). Consider the family

$$\mathcal{F} := \big\{(W_\alpha, \le_\alpha),\ \alpha \in I\big\} \tag{2}$$

of all well ordered sets $(W_\alpha, \le_\alpha)$ satisfying $m := f(X) = \min_\alpha W_\alpha$ – the minimal element of W_α with respect to the ordering $\le_\alpha$, and

$$f(X \setminus S) = \min_\alpha (W_\alpha \setminus S) \in W_\alpha \setminus S \quad \text{for every proper segment} \quad S \quad \text{of} \quad (W_\alpha, \le_\alpha). \tag{P}$$

This means that $f(X \setminus S)$ is the first subsequent element in $(W_\alpha, \le_\alpha)$ following S. The equality in (P) can be rewritten in the form

$$S = [m, a)_\alpha := \big\{x \in W_\alpha :\ x <_\alpha a\big\}, \quad \text{where} \quad a := f(X \setminus S) \in W_\alpha \setminus S. \tag{P1}$$

Indeed, since S is a segment in $(W_\alpha, \le_\alpha)$, the inequality $a <_\alpha x$ for $x \in S$ brings to a contradiction: $a \in [m, x)_\alpha \subset S$, whereas $a \notin S$. Therefore, $x <_\alpha a$ for all $x \in S$, i.e. $S \subseteq [m, a)_\alpha$. On the other hand, from $a = \min_\alpha (W_\alpha \setminus S)$ it follows $[m, a)_\alpha \subseteq S$. Hence we have $S = [m, a)_\alpha$.

Let well ordered sets $(W_1, \leq_1)$ and $(W_2, \leq_2)$ satisfy (P). Consider the common segments S of both these ordered sets, such that the ordering $\leq_1$ agrees with $\leq_2$ on S. Then the union S_0 of all common segments is the maximal (by inclusion) common segment, and $S_0 \subseteq W_1 \cap W_2$. We claim that S_0 coincides with at least one of the sets W_1 or W_2. Indeed, suppose otherwise. Then S_0 is a proper segment in each of these two ordered sets, and from (P) it follows

$$a_0 := f(X \setminus S_0) = \min_k(W_k \setminus S_0) \in W_k \setminus S_0 \quad \text{for} \quad k = 1, 2, \quad \text{hence} \quad a_0 \in (W_1 \cap W_2) \setminus S_0.$$

By virtue of (P1), $S_0 = [m, a_0)_1 = [m, a_0)_2$. Then $S_0 \cup \{a_0\}$ is a common segment of $(W_1, \leq_1)$ and $(W_2, \leq_2)$, in contradiction with the maximality of S_0.

The previous argument shows that for every two well ordered sets in (2), one is the extension of another, with the same ordering on the smaller set, which is a segment in the larger set. Note that if $x \in W_{\alpha_1}$ and $y \in W_{\alpha_2}$, then both $x, y \in W_\alpha$ – the largest of W_{α_1} and W_{α_2}. This allows us to define the chain $(X_0, \leq)$, where

$$X_0 := \bigcup_{\alpha \in I} W_\alpha, \quad \text{and} \quad x \leq y \quad \text{if and only if} \quad x, y \in W_\alpha \quad \text{and} \quad x \leq_\alpha y \quad \text{for some} \quad \alpha. \tag{3}$$

Further, we claim that $(X_0, \leq)$ is well ordered. Let A be a non-empty subset of X_0. Then $A \cap W_\beta$ is non-empty for some β. Since $(W_\beta, \leq_\beta)$ is well ordered, there exists $a := \min_\beta(A \cap W_\beta) \in A \cap W_\beta \subseteq A$. For the proof of our claim, it suffices to show that $a = \min A$, i.e. $a \leq x$ for every $x \in A$, which in turn means that if $a, x \in A \cap W_\alpha$, then $a \leq_\alpha x$. If $x \in A \cap W_\alpha \cap W_\beta$, this is true because the ordering $\leq_\alpha$ agrees with $\leq_\beta$, and a is the minimal element for a larger set $A \cap W_\beta$. In the remaining case $x \in A \cap (W_\alpha \setminus W_\beta)$, the set W_β is a proper segment of $(W_\alpha, \leq_\alpha)$. By virtue of (P1) and (P), we have

$$a \in W_\beta = [m, b)_\alpha, \quad \text{where} \quad b := f(X \setminus W_\beta) = \min_\alpha(W_\alpha \setminus W_\beta),$$

so that $a <_\alpha b \leq_\alpha x$. Thus we have $a \leq x$ for every $x \in A$, i.e. $a = \min A$, and $(X_0, \leq)$ is well ordered.

If S is a proper segment of $(X_0, \leq)$, then the set $W_\alpha \setminus S$ is non-empty for some α, and S is a proper segment of $(W_\alpha, \leq_\alpha)$ for some α. According to (P), we have

$$a := f(X \setminus S) = \min_\alpha(W_\alpha \setminus S) = \min(X_0 \setminus S). \tag{4}$$

Here the last equality follows from the facts that (i) the ordering $\leq$ agrees with $\leq_\alpha$ on $W_\alpha \setminus S$, and (ii) if $x \in X_0 \setminus W_\alpha$, then $x \in W_\beta \setminus W_\alpha$, where W_α is a proper segment of W_β, so that $a \leq x$ (which is the same as $a \leq_\beta x$) holds true automatically.

The property (4) allows us to include $(X_0, \leq)$ into the family $\mathcal{F}$ in (2). Finally, it remains to note that $X_0 = X$, because otherwise one can compose a larger well ordered set $(X_0 \cup \{a_0\}, \leq)$ by taking $a_0 := f(X \setminus X_0) \notin X_0$ as the subsequent element following X_0. This extended set also belongs to the family $\mathcal{F}$, which contradicts to the choice of X_0 in (3). Theorem is proved. $\square$

Remark 2. In the opposite direction, the Axiom of Choice follows from the Well Ordering Principle, simply by taking $f(A) := m(A)$ in (2).

Theorem 3 (**The Hausdorff Maximal Principle**). *Every chain* $(A, \leq)$ *in a partially ordered set* $(X, \leq)$ *is contained in a maximal chain* $(L, \leq)$. *In particular, maximal chains exist, because one can always start with a single point set* $A := \{a\}$, *which satisfies* $a \leq a$.

Proof. If $A = X$, then there is nothing to prove. In the contrary case, the set $X_0 := X \setminus A$ is non-empty, and by Theorem 1, there is an ordering $\leq_0$ (which has no relation to $\leq$) such that $(X_0, \leq_0)$ is a well ordered set.

Denote $m_0 := \min_0 X_0 \in X_0$ – the minimal element in X_0 with respect to the ordering $\leq_0$. There are two possible cases: (i) m_0 is comparable with every element $x \in A$, i.e. we have either $m_0 \leq x$ or $x \leq m_0$; and the contrary case (ii) m_0 is not comparable with some of $x \in A$. In other words, we have either (i) $(A \cup \{m_0\}, \leq)$ is a chain, or (ii) $(A \cup \{m_0\}, \leq)$ is not a chain.

Following the well ordering $\leq_0$, we can proceed by induction, deciding for every $x \in X_0$ whether or not it should be included into a chain $(L_x, \leq)$, which appears as an extension of the original chain $(A, \leq)$. At the initial step, for $x = m_0$, we have either (i) $L_{x_0} = A \cup \{m_0\}$ or (ii) $L_{x_0} = A$. We denote

$$L'_x := \bigcup \big\{ L_y : \quad y \in X_0, \quad y <_0 x \big\},$$

assuming that all L_y in this expression are already defined.

Let $B \subseteq X_0$ be the set of all elements b such that for all $x \in X_0$ satisfying $m_0 \leq_0 x \leq_0 b$, the chains $(L_x, \leq)$ are uniquely defined and satisfy

$$(i) \quad L_x := L'_x \cup \{x\} \quad \text{if} \quad \big(L'_x \cup \{x\}, \leq \big) \quad \text{is a chain,} \qquad (ii) \quad L_x := L'_x \quad \text{otherwise.} \tag{5}$$

We claim that $B = X_0$. Indeed, otherwise $X_0 \setminus B$ is non-empty, and $\exists a_0 := \min_0(X_0 \setminus B) \in X_0 \setminus B$. Then L'_{a_0} is the union of L_y over $y \in B$, hence one can uniquely define L_{a_0} according to (5), and we must have $a_0 \in B$. This contradiction proves the claim.

Finally, let L be the union of L_x over $x \in X_0$. We need to show that the arbitrary $x, y \in L$ are comparable, i.e. $x \leq y$ or $y \leq x$. If at least one of x or y belongs to A, this follows from $A \subseteq A_x$ for all $x \in X_0$. In the remaining case $x, y \in L \setminus A$, we can assume for certainty that $y <_0 x$. Then by construction in (5),

$$y \in L_y \subseteq L'_x \subseteq L_x, \quad \text{and} \quad x \in L_x.$$

Since both $x, y \in L_x$, they are comparable. Thus $(L, \leq)$ is a chain.

This chain is maximal, because if $x \notin L$, then $x \in X_0$ and $x \notin L_x$. Then by virtue of (4), x is not comparable with some elements of $L'_x \subseteq L$. Theorem is proved. □