Math 8601: Real Analysis: Fall 2015

Appendix B. Lebesgue measure

In this note, we partially follow the book:
A. N. Kolmogorov, S. V. Fomin, "Elements of the theory of functions and functional analysis".

Set $\Omega := [0,1]^d$. Consider **rectangles** $I := I_1 \times \cdots \times I_d$, where I_k for each k is one of the subsets

$$(a_k, b_k), \quad (a_k, b_k], \quad [a_k, b_k), \quad \text{or} \quad [a_k, b_k] \quad \text{of} \quad [0,1].$$

The **measure** of such a rectangle,

$$m(I) := \prod_{k=1}^{d} (b_k - a_k).$$

Definition 1. *Elementary* sets $\mathcal{E}$ are finite unions of rectangles in Ω.

$\mathcal{E}$ is closed with respect to *finite* number of operations $\cup, \cap, (\cdot)^c$, i.e. $\mathcal{E}$ is an **algebra** (a **field**). Each elementary set A can be represented as a finite union

$$A = \bigcup_{i=1}^{n} A_i \quad \text{of } \textbf{disjoint} \text{ rectangles.}$$

Then the measure $m(A) = \sum_i m(A_i)$ does not depend on representation (some of A_i can be subdivided into smaller rectangles). Note that $m(A)$ is an **additive** function on $\mathcal{E}$, i.e.

$$\text{if} \quad A = \bigcup_{i=1}^{n} A_i \quad \text{with} \quad A, A_i \in \mathcal{E} \quad \text{and} \quad \{A_i\} \quad \text{are disjoint, then} \quad m(A) = \sum_{i=1}^{n} m(A_i).$$

Theorem 2. *If $A, A_1, A_2, \dots$ are elementary sets, and*

$$A \subseteq \bigcup_{i=1}^{\infty} A_i, \quad \text{then} \quad m(A) \le \sum_{i=1}^{\infty} m(A_i).$$

Proof. $\forall \varepsilon > 0$, $\exists$ a compact $F \in \mathcal{E}$ and an open $G_i \in \mathcal{E}$, such that

$$F \subseteq A, \qquad m(A \setminus F) < \varepsilon; \qquad A_i \subseteq G_i, \qquad m(G_i \setminus A_i) < \frac{\varepsilon}{2^i}.$$

We have

$$F \subseteq A \subseteq \bigcup_{i=1}^{\infty} A_i \subseteq \bigcup_{i=1}^{\infty} G_i.$$

Since F is compact, $\exists N$ such that

$$F \subseteq \bigcup_{i=1}^{N} G_i, \qquad m(F) \le \sum_{i=1}^{N} m(G_i). \tag{1}$$

Since m is additive, we have

$$\begin{aligned} m(A) &= m(F) + m(A \setminus F) < m(F) + \varepsilon, \\ m(G_i) &= m(A_i) + m(G_i \setminus A_i) < m(A_i) + \frac{\varepsilon}{2^i}. \end{aligned}$$

Together with (1), these inequalities imply

$$m(A) \le \varepsilon + \sum_{i=1}^{N} \Big(m(A_i) + \frac{\varepsilon}{2^i} \Big) < 2\varepsilon + \sum_{i=1}^{\infty} m(A_i).$$

The desired inequality $m(A) \le \sum_i m(A_i)$ follows by taking $\varepsilon \to 0^+$. ☐

Definition 3. The **outer measure** of an **arbitrary** subset $A \subseteq \Omega := [0,1]^d$,

$$m^*(A) := \inf \left\{ \sum_{n=1}^{\infty} m(A_n) : \quad A \subseteq \bigcup_{n=1}^{\infty} A_n, \quad A_n - \text{rectangles} \right\}.$$

Obviously, $m^*(\emptyset) = 0$ and from $A \subseteq B$ it follows $m^*(A) \le m^*(B)$.

Theorem 4 (subadditivity)**.** *For arbitrary sets $A, A_1, A_2, \ldots \subseteq \Omega := [0,1]^d$,*

$$A \subseteq \bigcup_{n=1}^{\infty} A_n \quad \Longrightarrow \quad m^*(A) \le \sum_{n=1}^{\infty} m^*(A_n).$$

Proof. Fix $\varepsilon > 0$. By Definition 3, $\forall n \ge 1$, $\exists$ rectangles $A_{n,k}$, $k \ge 1$, such that

$$A_n \subseteq \sum_{k=1}^{\infty} A_{n,k} \quad \text{with} \quad \sum_{k=1}^{\infty} m\big(A_{n,k}\big) < m^*\big(A_n\big) + \frac{\varepsilon}{2^n}.$$

We can re-arrange the countable family of sets $\big\{A_{n,k} : \ n,k \ge 1\big\}$ into one sequence $\{B_j : \ j \ge 1\}$, so that

$$A \subseteq \sum_{n=1}^{\infty} A_n \subseteq \sum_{n,k=1}^{\infty} A_{n,k} = \sum_{j=1}^{\infty} B_j,$$

and

$$m^*(A) \le \sum_{j=1}^{\infty} m(B_j) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} m(A_{n,k}) \le \sum_{n=1}^{\infty} \Big(m^*(A_n) + \frac{\varepsilon}{2^n} \Big) = \varepsilon + \sum_{n=1}^{\infty} m^*(A_n).$$

Since $\varepsilon > 0$ is arbitrary, we have

$$m^*(A) \le \sum_n m^*(A_n).$$

☐

Remark 5. Note that $m^* = m$ on a algebra $\mathcal{E}$ of all elementary sets $A \subseteq \Omega = [0,1]^d$. Indeed, each elementary set

$$A = \bigcup_{n=1}^{N} A_n \quad - \text{a finite union of disjoint rectangles.}$$

Take $A_n = \emptyset$ for $n = N+1, N+2, \ldots$. Then $A = \bigcup_{n=1}^{\infty} A_n$, so that by definition of $m^*(A)$ and finite additivity,

$$m^*(A) \le \sum_{n=1}^{\infty} m(A_n) = \sum_{n=1}^{N} m(A_n) = m(A).$$

By Theorem 2, we also have $m(A) \le m^*(A)$ so we must have $m^*(A) = m(A)$, $\forall A \in \mathcal{E}$.

Further, any open set in $\Omega := [0,1]^d$ is a finite or countable union of rectangles. Therefore, the Borel σ-algebra

$$\mathcal{B} := \sigma(\mathcal{O}) = \sigma(\mathcal{E}), \quad \text{where} \quad \mathcal{O}\text{– all open sets in } \Omega.$$

Our goal is to extend $m(A)$ from $\mathcal{E}$ to a σ-algebra $\mathcal{F} \supseteq \mathcal{E}$ (then $\mathcal{F} \supseteq \mathcal{B} := \sigma(\mathcal{E})$) in such a way that the extended measure is σ-additive on $\mathcal{B}$, i.e. the additivity property is extended to countable union of disjoint sets. It turns out that one can take $\mathcal{F}$ = Lebesgue measurable sets in the following definition, and $m = m^*$ on $\mathcal{F}$. Note that there are other equivalent definitions.

Definition 6. A set $A \subseteq \Omega := [0,1]^d$ is **Lebesgue measurable** if $\forall \varepsilon > 0$, $\exists$ an elementary set $B \in \mathcal{E}$ such that

$$m^*(A \triangle B) < \varepsilon, \quad \text{where} \quad A \triangle B := (A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (AB) - \tag{2}$$

the **symmetric difference** of A and B.

Lemma 7.

$$|m^*(A) - m^*(B)| \le m^*(A \triangle B).$$

Proof. Since $A \subseteq B \cup (A \triangle B)$, we have by subadditivity

$$m^*(A) \le m^*(B) + m^*(A \triangle B).$$

By symmetry, one can interchange A and B. □

Lemma 8. *For arbitrary sets A and B, we have*

$$A \triangle B = A^c \triangle B^c. \tag{3}$$

Moreover, for arbitrary families of sets $\{A_\alpha\}$ and $\{B_\alpha\}$, $\alpha \in I$ – a common set of indices, we have

$$\Big(\bigcup_\alpha A_\alpha\Big) \triangle \Big(\bigcup_\alpha B_\alpha\Big) \quad \subseteq \quad \bigcup_\alpha (A_\alpha \triangle B_\alpha), \tag{3a}$$

$$\Big(\bigcap_\alpha A_\alpha\Big) \triangle \Big(\bigcap_\alpha B_\alpha\Big) \quad \subseteq \quad \bigcup_\alpha (A_\alpha \triangle B_\alpha). \tag{3b}$$

Proof. The equality (3) is obvious. For the proof of (3a), note that

$$x \in \Big(\bigcup_\alpha A_\alpha\Big) \setminus \Big(\bigcup_\alpha B_\alpha\Big) \quad \Longrightarrow \quad x \in A_\alpha \setminus B_\alpha \subseteq A_\alpha \triangle B_\alpha \quad \text{for some} \quad \alpha$$
$$\Longrightarrow \quad \Big(\bigcup_\alpha A_\alpha\Big) \setminus \Big(\bigcup_\alpha B_\alpha\Big) \quad \subseteq \quad \bigcup_\alpha (A_\alpha \triangle B_\alpha).$$

Interchanging A_α and B_α, we get (3a).

Using (3), we can now derive (3b) from (3a) as follows:

$$\Big(\bigcap_\alpha A_\alpha\Big) \triangle \Big(\bigcap_\alpha B_\alpha\Big) = \Big(\bigcap_\alpha A_\alpha\Big)^c \triangle \Big(\bigcap_\alpha B_\alpha\Big)^c$$
$$= \Big(\bigcup_\alpha A_\alpha^c\Big) \triangle \Big(\bigcup_\alpha B_\alpha^c\Big) \quad \subseteq \quad \bigcup_\alpha (A_\alpha^c \triangle B_\alpha^c) = \bigcup_\alpha (A_\alpha \triangle B_\alpha).$$

□

Theorem 9. (a). *The family $\mathcal{F}$ of all Lebesgue measurable sets is an algebra.*
(b). *m^* is additive on $\mathcal{F}$.*

Proof. **(a).** By virtue of (3), from $A \in \mathcal{F}$ it follows that $A^c \in \mathcal{F}$. Therefore, it remains to show that from $A_1, A_2 \in \mathcal{F}$ it follows that $A := A_1 \cup A_2 \in \mathcal{F}$.

Fix $\varepsilon > 0$ and choose $B_1, B_2 \in \mathcal{E}$ such that

$$m^*(A_1 \triangle B_1) < \varepsilon, \qquad m^*(A_2 \triangle B_2) < \varepsilon. \tag{4}$$

Note that by (3a),

$$(A_1 \cup A_2) \triangle (B_1 \cup B_2) \quad \subseteq \quad (A_1 \triangle B_1) \cup (A_2 \triangle B_2).$$

Using subadditivity and monotonicity of m^*, we obtain for $A := A_1 \cup A_2,\ B := B_1 \cup B_2$:

$$m^*(A \triangle B) \le m^*(A_1 \triangle B_1) + m^*(A_2 \triangle B_2) < \varepsilon + \varepsilon = 2\varepsilon. \tag{5}$$

This means $A := A_1 \cup A_2 \in \mathcal{F}$. By taking $(A_1 \cap A_2)^c = A_1^c \cup A_2^c$, we also get $A_1 \cap A_2 \in \mathcal{F}$.

(b). By induction, it suffices to check the additivity for two disjoint sets:

$$A_1, A_2 \in \mathcal{F}, \qquad A_1 A_2 := A_1 \cap A_2 = \emptyset.$$

Fix $\varepsilon > 0$ and choose $B_1, B_2 \in \mathcal{E}$ as in (4). Set $A := A_1 \cup A_2,\ B := B_1 \cup B_2$. Note that $A_1 A_2 = \emptyset$ and from (3b) it follows

$$B_1 B_2 = (A_1 A_2) \triangle (B_1 B_2) \quad \subseteq \quad (A_1 \triangle B_1) \cup (A_2 \triangle B_2).$$

As in (5), we get (remember that $m^* = m$ on the family $\mathcal{E}$ which contains $B_1, B_2, B_1 \triangle B_2, B_1 B_2$):

$$m^*(A \triangle B) < 2\varepsilon, \qquad m(B_1 B_2) < 2\varepsilon.$$

By Lemma 7,

$$|m^*(A_1) - m(B_1)| < \varepsilon, \qquad |m^*(A_2) - m(B_2)| < \varepsilon,$$

and

$$m^*(A) \ge m(B) - m^*(A \triangle B) > m(B) - 2\varepsilon. \tag{6}$$

Uisng additivity of m on $\mathcal{E}$, we obtain

$$\begin{aligned} m(B) &= m(B_1 \cup B_2) = m(B_1) + m(B_2) - m(B_1 B_2) \\ &> m(B_1) + m(B_2) - 2\varepsilon > m^*(A_1) + m^*(A_2) - 4\varepsilon. \end{aligned}$$

By virtue of (6),

$$m^*(A_1 \cup A_2) = m^*(A) > m^*(A_1) + m^*(A_2) - 6\varepsilon.$$

Since $\varepsilon > 0$ can be arbitrarily small, this implies

$$m^*(A_1 \cup A_2) \ge m^*(A_1) + m^*(A_2).$$

The opposite inequality is always true by subadditivity.

Theorem is proved. $\square$

Theorem 10. (a). *$\mathcal{F}$ is a σ-algebra.*

(b). *m^* is σ-additive on $\mathcal{F}$.*

Proof. **(a).** Let $A := \bigcup_{n=1}^{\infty} A_n$. One can always assume that $\{A_n\}$ are disjoint, because otherwise, one can replace A_n by

$$A'_n := A_n \setminus \Big(\bigcup_{k=1}^{n-1} A_k \Big).$$

We can write

$$A = C_N \cup D_N, \quad \text{where} \quad C_N := \bigcup_{n \le N} A_n, \qquad D_N := \bigcup_{n > N} A_n.$$

Then by additivity of m^*,

$$\sum_{n=1}^{N} m^*(A_n) = m^*(C_N) \le m^*(A) \le m^*(\Omega) = 1.$$

Fix $\varepsilon > 0$ and choose N such that

$$\sum_{n>N} m^*(A_n) < \varepsilon, \quad \text{so that by subadditivity} \quad m^*(D_N) < \varepsilon.$$

By Theorem 9(a), $\exists B_N \in \mathcal{E}$ such that

$$m^*(C_N \triangle B_N) < \varepsilon.$$

Then by (3a),

$$A \triangle B_N = (C_N \cup D_N) \triangle (B_N \cup \emptyset) \quad \subseteq \quad (C_N \triangle B_N) \cup D_N,$$

which in turn implies

$$m^*(A \triangle B_N) \le m^*(C_N \triangle B_N) + m^*(D_N) < 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, we have $A \in \mathcal{F}$.

(b). Let $A := \bigcup_{n=1}^{\infty} A_n$ with disjoint A_n. In the previous part (a), we proved that $A \in \mathcal{F}$. Using the additivity of m^*, in the previous notations we get:

$$\sum_{n=1}^{N} m^*(A_n) = m^*(C_N) \le m^*(A), \quad \forall N \ge 1.$$

Therefore,

$$\sum_{n=1}^{\infty} m^*(A_n) \le m^*(A).$$

The opposite inequality is always true by subadditivity.

Theorem is proved. $\square$

This completes the construction of the Lebesgue measure $m^* = m$ on Lebesgue sets in $\Omega := [0,1]^d$. Since the boundary has zero measure, nothing changes if Ω is replaced by $\Omega_0 := [0,1)^d$. One can use decomposition

$$\mathbb{R}^d = \bigcup_i (i + \Omega_0),$$

where $i = (i_1, \ldots, i_d)$ – vectors with integer components, in order to define

$$m(A) := \sum_i m\big(A \cap (i + \Omega_0)\big) \quad \text{for} \quad A \subseteq \mathbb{R}^d.$$