# Appendix C. Extension of continuous functions

Let $f(x)$ be a continuous function on a compact set $K \subset \mathbb{R}^d$. Then it is uniformly continuous on $K$, i.e. its **modulus of continuity**

$$\omega(\rho) := \sup\Big\{|f(x)-f(y)| : \quad x, y \in K,\ |x-y| \le \rho\Big\} \searrow 0 \quad \text{as} \quad \rho \searrow 0. \tag{1}$$

**Lemma 1.** *If $K$ is* **convex***, then $\omega(\rho)$ is* **subadditive***, i.e*

$$\omega(\rho_1+\rho_2) \le \omega(\rho_1)+\omega(\rho_2) \quad \text{for} \quad \rho_1, \rho_2 \ge 0. \tag{2}$$

*Proof.* For arbitrary $x, y \in K$ with $|x-y| \le \rho_1+\rho_2$, the segment $[x,y]$ lies in $K$, and there is a point $z \in [x,y]$ such that $|x-z| \le \rho_1$, $|y-z| \le \rho_2$. Therefore,

$$|f(x)-f(y)| \le |f(x)-f(z)| + |f(z)-f(y)| \le \omega(\rho_1)+\omega(\rho_2),$$

and (2) follows. $\square$

**Remark 2.** *The property (2) fails in general if $K$ is not convex. In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, an easy example is*

$$f(x,y) := \theta \quad \text{on} \quad K := \{1 \le r \le 2, \quad \varepsilon \le \theta \le 2\pi - \varepsilon\} \quad \text{with a small} \quad \varepsilon > 0.$$

**Lemma 3.** *For any continuous function $f$ on a compact $K \subset \mathbb{R}^d$ with the modulus of continuity $\omega(\rho)$ in (1), the function*

$$\overline{\omega}(\rho) := \sup_{s \ge 1} \frac{\omega(\rho s)}{s}, \qquad \rho \ge 0, \tag{3}$$

*satisfies the properties:*
**(i)** $\overline{\omega}(\rho) \ge \omega(\rho)$,
**(ii)** $\overline{\omega}(\rho) \searrow 0$ *as* $\rho \searrow 0$,
**(iii)** $\overline{\omega}(\rho)$ *is subadditive, i.e. it satisfies (2).*

*Proof.* **(i)** is obvious.

**(ii)** Since $\omega(\rho)$ is non-decreasing, the function $\overline{\omega}(\rho)$ is also non-decreasing for $\rho \ge 0$. We also have

$$\omega(\rho) \le C_0 := 2\sup_K |f|, \quad \text{and} \quad \overline{\omega}(\rho) \le C_0.$$

For an arbitrary $A > 1$, we can write

$$\overline{\omega}(\rho) = \max\left\{ \sup_{A \geq s \geq 1} \frac{\omega(\rho s)}{s}, \quad \sup_{s \geq A} \frac{\omega(\rho s)}{s} \right\} \leq \max\left\{ \omega(A\rho),\ \frac{C_0}{A} \right\}.$$

This implies

$$\limsup_{\rho \to 0^+} \ \overline{\omega}(\rho) \leq \frac{C_0}{A},$$

and since $A > 1$ can be chosen arbitrarily large, we have $\overline{\omega}(\rho) \searrow 0$ as $\rho \searrow 0$.

**(iii)** This part follows directly from subadditivity of $\omega$ in (2) $\square$

**Theorem 4.** *Let $f$ be a continuous function on a compact $K \subset \mathbb{R}^d$. Then the function*

$$F(x) := \inf_{y \in K} \Big[ f(y) + \overline{\omega}(|x - y|) \Big], \qquad x \in \mathbb{R}^d, \tag{4}$$

*where $\overline{\omega}(\rho)$ is defined in (3), satisfies the properties:*
**(i)** *$F \equiv f$ on $K$,*
**(ii)** *$F$ provides a continuous extension of $f$ from $K$ to $\mathbb{R}^d$, and*

$$|F(x_1) - F(x_2)| \leq \overline{\omega}(|x_1 - x_2|),\ \forall x_1, x_2 \in \mathbb{R}^d. \tag{5}$$

*Proof.* **(i)** Note that $\forall x, y \in K$, we have

$$f(x) \leq f(y) + \omega(|x - y|) \leq f(y) + \overline{\omega}(|x - y|),$$

with the equality at $y = x$. Therefore, $f(x) = F(x),\ \forall x \in K$.

**(ii)** By monotonicity and subadditivity of $\overline{\omega}(\rho)$, we have $\forall x_1, x_2, y \in K$:

$$\overline{\omega}(|x_1 - y|) \leq \overline{\omega}\big(|x_1 - x_2| + |x_2 - y|\big) \leq \overline{\omega}(|x_1 - x_2|) + \overline{\omega}(|x_2 - y|).$$

Hence the function $F(x)$ in (4) satisfies

$$F(x_1) \leq \overline{\omega}(|x_1 - x_2|) + F(x_2).$$

Interchanging $x_1$ and $x_2$, we get the desired property (5). $\square$