Math 8601: Real Analysis: Fall 2015

Chapter 1: Measures

This note is supplementary to the textbook:
G.B. Folland, REAL ANALYSIS. Modern Techniques and Their Applications. 2nd Edition.
We keep same notations for same or similar statements, for which our proofs deviates from the text, and add some material from other sources. Statements or formulas, which are different or not numbered in the textbook, are labeled by "I-...". We also use label "B-..." in references to Appendix B, e.g. Theorem B-4 means Theorem 4 in Appendix B.

Definition I-1. A measure μ on a measurable space $(X, \mathcal{M})$ is called **complete** if

$$F \subseteq E \in \mathcal{M}, \quad \mu(E) = 0 \quad \Longrightarrow \quad F \in \mathcal{M}.$$

This definition is a bit misleading: it would be more natural to call σ-algebra $\mathcal{M}$ complete, but we cannot break this tradition.

The following formulation is equivalent to that in the textbook.

1.9. Theorem. *For an arbitrary measure space $(X, \mathcal{M}, \mu)$, denote*

$$\overline{\mathcal{M}} := \big\{E \subseteq X : \quad \exists E_{1,2} \in \mathcal{M} \quad \text{such that} \quad E_1 \subseteq E \subseteq E_2 \quad \text{and} \quad \mu(E_2 \setminus E_1) = 0\big\}. \tag{I-1}$$

Then $\overline{\mathcal{M}}$ is a σ-algebra containing $\mathcal{M}$, and there is a unique measure $\overline{\mu}$ on $\overline{\mathcal{M}}$ such that $\overline{\mu} = \mu$ on $\mathcal{M}$. The measure space $(X, \overline{\mathcal{M}}, \overline{\mu})$ is called the **completion** *of $(X, \mathcal{M}, \mu)$.*

Proof. If $E_1 \subseteq E \subseteq E_2$, then $E_2^c \subseteq E^c \subseteq E_1^c$. Since $E_1^c \setminus E_2^c = E_2 \setminus E_1$, from $E \in \overline{\mathcal{M}}$ it follows $E^c \in \overline{\mathcal{M}}$.

Next, let a sequence $\big\{E^{(j)}\big\} \subseteq \overline{\mathcal{M}}$. Then

$$\begin{aligned} &\exists E_{1,2}^{(j)} \in \mathcal{M} \quad \text{such that} \quad E_1^{(j)} \subseteq E^{(j)} \subseteq E_2^{(j)} \quad \text{and} \quad \mu\big(E_2^{(j)} \setminus E_1^{(j)}\big) = 0 \\ &\Longrightarrow \quad E_1 := \bigcup_j E_1^{(j)} \subseteq E := \bigcup_j E^{(j)} \subseteq E_2 := \bigcup_j E_2^{(j)}. \end{aligned}$$

Here $E_{1,2} \in \mathcal{M}$, and

$$\mu(E_2 \setminus E_1) \le \mu\bigg(\bigcup_j \big(E_2^{(j)} \setminus E_1^{(j)}\big)\bigg) \le \sum_j \mu\big(E_2^{(j)} \setminus E_1^{(j)}\big) = 0.$$

This means that $E \in \overline{\mathcal{M}}$. Since $\overline{\mathcal{M}}$ is closed with respect to operations of taking complements and countable unions, it is a σ-algebra.

It is easy to verify that $\overline{\mu}(E) := \mu(E_1)$ in (I-1) is a unique σ-additive extension of μ from $\mathcal{M}$ to $\overline{\mathcal{M}}$. $\square$

Definition I-2. An **outer measure** on a nonempty set X is a function $\mu^* : 2^X \to [0, \infty]$ such that
(i) $\mu^*(\emptyset) = 0$,
(ii) (monotonicity) $A \subseteq B \quad \Longrightarrow \quad \mu^*(A) \le \mu^*(B)$,
(iii) (subadditivity)

$$\mu^*\bigg(\bigcup_{j=1}^{\infty} A_j\bigg) \le \sum_{j=1}^{\infty} \mu^*(A_j). \tag{I-2}$$

A set $A \subseteq X$ is **μ^*-measurable** if

$$\mu^*(E) = \mu^*(EA) + \mu^*(EA^c), \qquad \forall E \subseteq X. \tag{I-3}$$

1.11. Carathéodory's Theorem. *The collection $\mathcal{M}$ of all μ^*-measurable sets is a σ-algebra, and μ^* is a complete measure on $\mathcal{M}$.*

Proof. **(i).** Obviously, $A \in \mathcal{M} \implies A^c \in \mathcal{M}$.

(ii). Next we show that $\mathcal{M}$ is an algebra. By induction, it suffices to verify that from $A_1, A_2 \in \mathcal{M}$ it follows $A := A_1 \cup A_2 \in \mathcal{M}$. Note that X is the union of four disjoint sets

$$C_1 := A_1 A_2, \quad C_2 := A_1 A_2^c, \quad C_3 := A_1^c A_2, \quad \text{and} \quad C_4 := A_1^c A_2^c. \tag{I-4}$$

Since $A_1 \in \mathcal{M}$, for an arbitrary $E \subseteq X$ we have

$$\mu^*(E) = \sum_{k=1}^{2} \mu^*(E_k), \quad \text{where} \quad E_1 := EA_1, \quad E_2 := A_1^c.$$

Since $A_2 \in \mathcal{M}$, we also have

$$\mu^*(E_k) = \mu^*(E_k A_2) + \mu^*(E_k A_2^c) \quad \text{for} \quad k = 1, 2.$$

From these equalities it follows

$$\mu^*(E) = \sum_{k=1}^{2} \Big(\mu^*(E_k A_2) + \mu^*(E_k A_2^c) \Big) = \sum_{j=1}^{4} \mu^*(EC_j). \tag{I-5}$$

Now note that $A := A_1 \cup A_2 = C_1 \cup C_2 \cup C_3$, and $A^c = A_1^c A_2^c = C_4$. Using subadditivity twice and then (I-5), we derive

$$\mu^*(E) \le \mu^*(EA) + \mu^*(EA^c) \le \sum_{j=1}^{4} \mu^*(EC_j) = \mu^*(E).$$

Since the left and right sides coincide, we arrive at (I-3), i.e. $A := A_1 \cup A_2 \in \mathcal{M}$, and $\mathcal{M}$ is an algebra.

We include the following lemma inside of the proof. Note that we first get σ-additivity of μ^* on the **algebra** $\mathcal{M}$, and then show that $\mathcal{M}$ is actually σ-algebra. This lemma is formulated in Exercise 17 on p. 32.

Lemma I-3. *Let $\{A_j\}$ be a sequence of disjoint sets in $\mathcal{M}$. Then*

$$\mu^*(EA) = \sum_{j=1}^{\infty} \mu^*(EA_j), \quad E \subseteq X, \quad \textit{where} \quad A := \bigcup_{j=1}^{\infty} A_j. \tag{I-6}$$

In particular (for $E = X$) we have σ-additivity of μ^* on $\mathcal{M}$.

Proof. First consider the case of two sets A_1 and A_2, i.e. $A_j = \emptyset$ for $j \ge 3$. Then in (I-4)

$$C_1 := A_1 A_2 = \emptyset, \quad C_2 := A_1 A_2^c = A_1, \quad C_3 := A_1^c A_2 = A_2, \quad \text{and} \quad C_4 := A_1^c A_2^c = (A_1 \cup A_2)^c.$$

Now (I-5) can be rewritten as

$$\mu^*(E) = \mu^*(EA_1) + \mu^*(EA_2) + \mu^*\big(E(A_1 \cup A_2)^c\big).$$

On the other hand, since $A_1 \cup A_2 \in \mathcal{M}$, we also have

$$\mu^*(E) = \mu^*\big(E(A_1 \cup A_2)\big) + \mu^*\big(E(A_1 \cup A_2)^c\big).$$

Comparing these two equalities, we see that

$$\mu^*\big(E(A_1 \cup A_2)\big) = \mu^*(EA_1) + \mu^*(E_2),$$

i.e. (I-6) holds true for two sets. By induction, this equality is extended to any finite number of disjoint sets $A_j \in \mathcal{M}$:

$$\mu^*(EB_n) = \sum_{j=1}^{n} \mu^*(EA_j), \quad \text{where} \quad B_n := \bigcup_{j=1}^{n} A_j \in \mathcal{M}. \tag{I-7}$$

By monotonicity, $\mu^*(EA) \geq \mu^*(EB_n)$. Since n is arbitrary, we have the inequality " $\geq$" between the two parts in (I-6). The opposite inequality is always true by subadditivity. Lemma is proved. $\square$

Now we complete the remaining parts of the proof of Carathéodory's Theorem.

(iii). Next we show that $\mathcal{M}$ is a σ-algebra, i.e. the set A in (I-6) belongs to $\mathcal{M}$. By the equality (I-6) and relations $B_n \subseteq A$, $B_n^c \supseteq A^c$, we get

$$\mu^*(E) = \mu^*(EB_n) + \mu^*(EB_n^c) \geq \sum_{j=1}^{n} \mu^*(EA_j) + \mu^*(EA^c).$$

By taking $n \to \infty$ and using subadditivity, we obtain

$$\mu^*(E) \geq \sum_{j=1}^{\infty} \mu^*(EA_j) + \mu^*(EA^c) \geq \mu^*(EA) + \mu^*(EA^c) \geq \mu^*(E).$$

Since both sides coincide, we must have the equality in (I-3), i.e. $A \in \mathcal{M}$. Hence $\mathcal{M}$ is a σ-algebra, and μ^* is a measure on $\mathcal{M}$.

(iv). Finally, if $A \subseteq X$ satisfies $\mu^*(A) = 0$, then

$$\mu^*(E) \leq \mu^*(EA) + \mu^*(EA^c) \leq \mu^*(A) + \mu^*(E) = \mu^*(E).$$

As in the previous arguments, we must have the equality in (I-3). Therefore, from $\mu^*(A) = 0$ it follows $A \in \mathcal{M}$, and by Definition I-1, μ^* is a **complete** measure on $\mathcal{M}$. Theorem is proved. $\square$

Definition I-4. A **premeasure** on an algebra $\mathcal{A} \subseteq 2^X$ is a function $\mu_0 : \mathcal{A} \to [0, \infty]$ such that
(i) $\mu_0(\emptyset) = 0$,
(ii) μ_0 is σ-additive on $\mathcal{A}$: if $A \in \mathcal{A}$ is a countable union of disjoint sets $A_j \in \mathcal{A}$, then

$$\mu_0(A) = \sum_{j=1}^{\infty} \mu_0(A_j).$$

It **induces** an outer measure on X:

$$\mu^*(E) := \inf \left\{ \sum_{j=1}^{\infty} \mu_0(A_j) : \quad A_j \in \mathcal{A}, \quad E \subseteq \bigcup_{j=1}^{\infty} A_j \right\}. \tag{1.12}$$

This is an outer measure according to Proposition 1.10. Its proof is similar to that of Theorem B-4.

Proposition 1.13. (a) *$\mu^* = \mu_0$ on $\mathcal{A}$;* **(b)** *every set in $\mathcal{A}$ is μ^*-measurable.*
See the textbook for the proof.

In the following theorem, the equivalence $(i) \iff (ii)$ is formulated Exercise 19 on p.32. In the case when $\mathcal{A} =$ (finite unions of intervals in $\mathbb{R}^1$), the implication $(i) \Longrightarrow (iii)$ is contained in Proposition 1.20.

Theorem I-5. *Assume that* $\mu_0(X) < \infty$*. Then the following are equivalent:*
(i) A *is* μ^**-measurable in the Carathéodory's sense:*

$$\mu^*(E) = \mu^*(EA) + \mu^*(EA^c), \qquad \forall E \subseteq X;$$

(ii) $\mu_0(X) = \mu^*(A) + \mu^*(A^c)$*;*
(iii) $\forall \varepsilon > 0, \quad \exists B \in \mathcal{A}$ *such that* $\mu^*(A\Delta B) < \varepsilon$*.*

Proof. **(i)** $\Longrightarrow$ **(ii)**. Take $E = X$ and note that $\mu^* = \mu_0$ on $\mathcal{A}$.

(ii) $\Longrightarrow$ **(iii)**. Let A satisfy the property (ii). Fix $\varepsilon > 0$. By (1.12),

$$\exists A_j \in \mathcal{A} \quad \text{such that} \quad A \subseteq \bigcup_{j=1}^{\infty} A_j \quad \text{and} \quad \sum_{j=1}^{\infty} \mu_0(A_j) < \mu^*(A) + \varepsilon.$$

Since $\mu^*(A) \le \mu_0(X) < \infty$, the last series converges, and for a large enough n we have

$$\sum_{j=n+1}^{\infty} \mu_0(A_j) < \varepsilon.$$

Then $A \subseteq B \cup B'$, where

$$B := \bigcup_{j=1}^{n} A_j \in \mathcal{A}, \quad \text{and} \quad B' := \bigcup_{j=n+1}^{\infty} A_j \quad \text{satisfies} \quad \mu^*(B') \le \sum_{j=n+1}^{\infty} \mu_0(A_j) < \varepsilon.$$

Therefore,

$$\mu^*(A) \le \mu_0(B) + \mu^*(B') \le \sum_{j=1}^{\infty} \mu_0(A_j) < \mu^*(A) + \varepsilon.$$

Similarly, $A^c \subseteq C \cup C'$ with $C \in \mathcal{A}$, $\mu^*(C') < \varepsilon$, and

$$\mu^*(A^c) \le \mu_0(C) + \mu^*(C') < \mu^*(A^c) + \varepsilon.$$

We have $X = (A \cup A^c) \subseteq (B \cup C) \cup B' \cup C^c$, where by additivity of μ_0 on $\mathcal{A}$,

$$\mu_0(B \cup C) = \mu_0(B) + \mu_0(C) - \mu_0(BC).$$

From these relations together with (ii) it follows

$$\begin{array}{rcl} \mu_0(X) & \le & \mu_0(B \cup C) + \mu^*(B') + \mu^*(C') \\ & \le & \mu_0(B) + \mu^*(B') + \mu_0(C) + \mu^*(C') - \mu_0(BC) \\ & < & \mu^*(A) + \mu^*(A^c) + 2\varepsilon - \mu_0(BC) \\ & = & \mu_0(X) + 2\varepsilon - \mu_0(BC). \end{array}$$

Therefore, $\mu_0(BC) < 2\varepsilon$. Finally, since

$$A \setminus B \subseteq B', \quad B \setminus A = BA^c \subseteq B(C \cup C') \subseteq (BC) \cup C',$$

we get

$$\mu^*(A\Delta B) \le \mu^*(A \setminus B) + \mu^*(B \setminus A) \le \mu^*(B') + \mu_0(BC) + \mu^*(C') < 4\varepsilon.$$

After replacing ε by $\varepsilon/4$, the desired property (iii) follows.

(iii) $\Longrightarrow$ (i). Let A satisfy (iii). Fix $\varepsilon > 0$ and choose $B \in \mathcal{A}$ such that $\mu^*(A\Delta B) < \varepsilon$. Note that

$$A \subseteq B \cup (A \setminus B) \subseteq B \cup (A\Delta B), \quad A^c \subseteq B^c \cup (A^c \Delta B^c) = B^c \cup (A\Delta B).$$

Hence for an arbitrary $E \subseteq X$,

$$\mu^*(EA) \le \mu^*(EB) + \mu^*(A\Delta B) < \mu^*(EB) + \varepsilon, \quad \mu^*(EA^c) < \mu^*(EB^c) + \varepsilon.$$

This implies

$$\mu^*(EA) + \mu^*(EA^c) < \mu^*(EB) + \mu^*(EB^c) + 2\varepsilon.$$

Further, by (1.12),

$$\exists \{A_j\} \subseteq \mathcal{A} \quad \text{such that} \quad E \subseteq \bigcup_{j=1}^{\infty} A_j \quad \text{and} \quad \sum_{j=1}^{\infty} \mu_0(A_j) < \mu^*(E) + \varepsilon.$$

By subadditivity of μ^* on 2^X and additivity of μ_0 on $\mathcal{A}$, we get

$$\begin{aligned} \mu^*(EA) + \mu^*(EA^c) &< \mu^*\Bigg(\bigcup_{j=1}^{\infty}(A_j B)\Bigg) + \mu^*\Bigg(\bigcup_{j=1}^{\infty}(A_j B^c)\Bigg) + 2\varepsilon \\ &\le \sum_{j=1}^{\infty} \mu_0(A_j B) + \sum_{j=1}^{\infty} \mu_0(A_j B^c) + 2\varepsilon \\ &= \sum_{j=1}^{\infty} \mu_0(A_j) + 2\varepsilon < \mu^*(E) + 3\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$\mu^*(E) \le \mu^*(EA) + \mu^*(EA^c) \le \mu^*(E),$$

and the property (i) follows. Theorem is proved. $\square$

In comparison with the textbook, the following theorem is restricted to the case of σ-finite μ_0. On the other hand, we consider extensions to the set of all μ-measurable sets, with contains $\mathcal{M} := \sigma(\mathcal{A})$. An alternative approach to part (ii) is outlined in Exercise 22a on p.32. Our proof may be longer because we incorporate a few useful facts. For example, the completion of $\mathcal{M}$ with respect to μ can be described as the family of all sets $A \subseteq X$ such that $\mu^*(A\Delta A_0) = 0$ for some $A_0 \in \mathcal{M} := \sigma(\mathcal{A})$.

1.14. Theorem. *Let μ_0 be a σ-finite premeasure on an algebra $\mathcal{A} \subseteq 2^X$. Denote $\mathcal{M} := \sigma(\mathcal{A})$, and $\mathcal{F}$ - the family of all μ^*-measurable sets.*
(i) *$\overline{\mu} := \mu^*|_{\mathcal{F}}$ is a unique measure on $\mathcal{F}$ such that $\overline{\mu} = \mu_0$ on $\mathcal{A}$.*
(ii) *If $\mu := \mu^*|_{\mathcal{M}}$, then $\overline{\mu}$ is the completion of μ.*

Proof. **(i).** In the textbook, this part is proved (with $\mathcal{M}$ instead of $\mathcal{F}$) by means of Carathéodory's theorem. Alternatively, one can adjust the construction in Appendix B by taking $(X, \mathcal{A}, \mu_0)$ in place of $(\Omega, \mathcal{E}, m)$.

First we assume that $\mu_0(X) < \infty$. Following Definition B-6 or part (iii) in Theorem I-5, introduce the family

$$\mathcal{F} := \Big\{A \subseteq X : \ \forall \varepsilon > 0, \ \exists B \in \mathcal{A} \quad \text{such that} \quad \mu^*(A\Delta B) < \varepsilon\Big\}. \tag{I-8}$$

Then the proofs of Theorems B-9 and B-10 remain valid, which state that $\mathcal{F}$ is a σ-algebra and $\overline{\mu} := \mu^*$ is σ-additive on $\mathcal{F}$. Moreover, since $\mathcal{A} \subseteq \mathcal{F}$, we also have $\mathcal{M} := \sigma(\mathcal{A}) \subseteq \mathcal{F}$.

If ν is another measure on $\mathcal{F}$, such that $\nu = \mu_0$ on $\mathcal{A}$, then

$$A \in \mathcal{F}, \quad A_j \in \mathcal{A}, \quad A \subseteq \bigcup_{j=1}^{\infty} A_j \quad \Longrightarrow \quad \nu(A) \le \sum_{j=1}^{\infty} \nu(A_j) = \sum_{j=1}^{\infty} \mu_0(A_j).$$

By (1.12), we get $\nu(A) \le \mu^*(A) =: \overline{\mu}(A)$, $\forall A \in \mathcal{F}$. Then also $\nu(A^c) \le \mu^*(A^c)$, hence

$$\mu_0(X) = \nu(X) = \nu(A) + \nu(A^c) \le \mu^*(A) + \mu^*(A^c) = \mu^*(X) = \mu_0(X).$$

This implies uniqueness, i.e. $\nu = \mu^* =: \overline{\mu}$ on $\mathcal{F}$.

In the case $\mu_0(X) = \infty$, the σ-finiteness of μ_0 means that

$$\exists E_j \nearrow X, \quad E_j \in \mathcal{A}, \quad \mu_0(E_j) < \infty \quad \text{for all} \quad j = 1, 2, \ldots.$$

Introduce the families $\mathcal{F}_j$ corresponding to E_j in place of X in (I-8). Then the family

$$\mathcal{F} := \Big\{ A \subseteq X : \quad AE_j \in \mathcal{F}_j, \quad \forall j = 1, 2, \ldots \Big\} \tag{I-9}$$

is a σ-algebra containing $\mathcal{A}$, and the unique extension of μ_0 from $\mathcal{A}$ to $\mathcal{F}$ is given by formula

$$\overline{\mu}(A) := \lim_{j \to \infty} \overline{\mu}(AE_j), \quad \forall A \in \mathcal{F}. \tag{I-10}$$

(ii). Let A be a set in the family $\mathcal{F}$ defined in (I-9) for σ-additive μ_0. Then

$$\forall \varepsilon > 0, \quad \forall j = 1, 2, \ldots, \quad \exists B_j \in \mathcal{A} \quad \text{such that} \quad \mu^*\big((AE_j)\Delta B_j\big) < \frac{\varepsilon}{2^j}.$$

Then

$$B := \bigcup_{j=1}^{\infty} B_j \quad \text{satisfies} \quad A\Delta B = \Big(\bigcup_{j=1}^{\infty} (AE_j) \Big) \Delta \Big(\bigcup_{j=1}^{\infty} B_j \Big) \subseteq \bigcup_{j=1}^{\infty} \Big((AE_j)\Delta B_j \Big).$$

Therefore,

$$\mu^*(A\Delta B) \le \sum_{j=1}^{\infty} \mu^*\Big((AE_j)\Delta B_j \Big) < \sum_{j=1}^{\infty} \frac{\varepsilon}{2^j} = \varepsilon.$$

In other words, for σ-additive μ_0, one needs to replace $\mathcal{A}$ by $\mathcal{M} := \sigma(\mathcal{A})$ in (I-8):

$$\mathcal{F} = \Big\{ A \subseteq X : \ \forall \varepsilon > 0, \ \exists B \in \mathcal{M} := \sigma(\mathcal{A}) \quad \text{such that} \quad \mu^*(A\Delta B) < \varepsilon \Big\}. \tag{I-11}$$

Further, one can choose a sequence $A_j \in \mathcal{A}$ such that $\mu^*(A\Delta A_j) \to 0$ as $j \to \infty$. Then by the triangle inequality, $\mu^*(A_j \Delta A_k) \to 0$ as $j, k \to \infty$. We can assume that the convergence is fast enough, so that one can apply the result in HW #4, Problem 3, with $(\mathcal{F}, \mu^*)$ in place of $\mathcal{M}, \mu)$. It guarantees that $\mu^*(A_0 \Delta A_j) \to 0$ as $j \to \infty$ for some $A_0 \in \mathcal{M}$. Since

$$\mu^*(A_0 \Delta A) \le \mu^*(A\Delta A_j) + \mu^*(A_0 \Delta A_j) \to 0 \quad \text{as} \quad j \to \infty,$$

we must have $\mu^*(A_0 \Delta A) = 0$, where $A_0 \in \mathcal{M}$. Moreover, by (1.12),

$$A\Delta A_0 \subseteq C_j \in \mathcal{M} \quad \text{with} \quad \mu(C_j) < \frac{1}{j}.$$

Then

$$A\Delta A_0 \subseteq C := \liminf_{j\to\infty} C_j := \bigcup_{n=1}^{\infty}\bigcap_{j=n}^{\infty} C_j \in \mathcal{M} \quad \text{with} \quad \mu(C)=0.$$

Hence $E_1 := A_0 \setminus C$ and $E_2 := A_0 \cup C$ belong to $\mathcal{M}$, and

$$E_1 \subseteq A_0 \setminus (A\Delta A_0) \subseteq A \subseteq A_0 \cup (A\Delta A_0) \subseteq E_2, \qquad \mu(E_2 \setminus E_1) \le \mu(C) = 0.$$

Since this is true for an arbitrary $A \in \mathcal{F}$, by (I-1) we have $\mathcal{F} = \overline{\mathcal{M}}$, and $(X, \mathcal{F}, \overline{\mu} = \mu^*)$ is the completion of $(X, \mathcal{M}, \mu)$. Theorem is proved. □

The following theorem generalizes Lemma 1.17 and Theorem 1.18 to the multidimensional case. In this theorem, $\mathcal{M}_\mu$ can be considered as the completion of the Borel σ-algebra $\mathcal{B}$ with respect to μ. We use same notation $\mu = \overline{\mu}$ for the corresponding unique extension of μ.

Theorem I-6. *Let μ be a Borel measure on $\mathbb{R}^d$ such that $\mu(K) < \infty$ $\forall$ compact $K \subseteq \mathbb{R}^d$, and let $\mathcal{M}_\mu$ denote the family of all μ-measurable sets in $\mathbb{R}^d$. Then*
(i) $\forall E \in \mathcal{M}_\mu$, $\forall \varepsilon > 0$

$$\exists \text{ closed } F \subseteq E \subseteq \text{ open } G \ \text{ with } \ \mu(G \setminus F) < \varepsilon. \tag{I-12}$$

(ii) *μ is* **regular**, *i.e.*

$$\mu(E) = \inf\Big\{\mu(G) :\ E \subseteq \text{ open } G\Big\} = \sup\Big\{\mu(K) :\ \text{compact } K \subseteq E\Big\}. \tag{I-13}$$

Proof. **(i).** Denote by W the family of all sets in $\mathcal{M}_\mu$ satisfying (1-12). We have

$$K_\delta := \prod_{j=1}^{\infty}[a_j+\delta, b_j] \nearrow Q := \prod_{j=1}^{\infty}(a_j, b_j], \quad G_\delta := \prod_{j=1}^{\infty}(a_j, b_j+\delta) \searrow Q \quad \text{as} \quad \delta \searrow 0. \tag{I-14}$$

Here $K_\delta \subseteq Q \subseteq G_\delta$, and by the continuity of μ (Theorem 1.8c,d), $\mu(G_\delta \setminus K_\delta) \to 0$ as $\delta \searrow 0$.
Hence W contains all the rectangles Q in (I-14).

Further, if $E \in \mathcal{M}_\mu$, then from (I-12) it follows

$$\text{closed } G^c \ \subseteq \ E^c \ \subseteq \ \text{open } F^c \quad \text{with} \quad \mu(F^c \setminus G^c) = \mu(G \setminus F) < \varepsilon,$$

i.e. $E^c \in \mathcal{M}_\mu$. Therefore, $\mathcal{M}_\mu$ contains the algebra of sets

$$\mathcal{A} := \Big\{A \subseteq \mathbb{R}^d :\ A \text{ or } A^c \text{ is a finite union of rectangles } Q \text{ in (I-14)}\Big\}. \tag{I-15}$$

Now consider a sequence $E_j \in W$, $j = 1, 2, \ldots$. For fixed $\varepsilon > 0$,

$$\exists \text{ closed } F_j \subseteq E_j \subseteq \text{ open } G_j \ \text{ with } \ \mu(G_j \setminus F_j) < \frac{\varepsilon}{2^j}.$$

This implies

$$F^* := \bigcup_{j=1}^{\infty} F_j \subseteq E = \bigcup_{j=1}^{\infty} E_j \subseteq \text{ open } G := \bigcup_{j=1}^{\infty} G_j \ \text{ with } \ \mu(G \setminus F^*) < \varepsilon,$$

where

$$(G \setminus F^*) \subseteq \bigcup_{j=1}^{\infty}(G_j \setminus F_j), \qquad \mu(G \setminus F^*) \le \sum_{j=1}^{\infty}\mu(G_j \setminus F_j) < \varepsilon. \tag{I-16}$$

In #HW 3, Problem 1, we can assume that $\mathbb{R}^d$ is represented as a union of **closed** cubes $\{I_k\}$, $k \ge 1$, with edge length 1, which are non-overlapping in the sense that their interiors $I_j^0 \cap I_k^0 = \emptyset$ for $j \ne k$. The closed sets

$$F_n^* := \bigcup_{j=1}^{n} F_j \nearrow F^*, \quad F_n^* I_k \nearrow F^* I_k \quad \text{as} \quad n \to \infty.$$

By continuity of μ, for fixed $\varepsilon > 0$, one can choose n_k such that

$$\mu\Big((F^* I_k) \setminus (F_{n_k}^*) I_k \Big) < \frac{\varepsilon}{2^k}.$$

Here we used the property $\mu(K) < \infty$ for $K = I_k$. Then by the above mentioned problem, the **closed** set

$$F := \bigcup_{k=1}^{\infty} (F_{n_k}^* I_k) \subseteq F^* = \bigcup_{k=1}^{\infty} (F^* I_k), \quad \text{and} \quad \mu(F^* \setminus F) \le \sum_{k=1}^{\infty} \mu\Big((F^* I_k) \setminus (F_{n_k}^*) I_k \Big) < \varepsilon.$$

Together with (I-16), this gives $\mu^*(G \setminus F) = \mu^*(G \setminus F^*) + \mu^*(F^* \setminus F) < 2\varepsilon$. Replacing ε by $\varepsilon/2$, we see that $E := \bigcup\limits_{j=1}^{\infty} E_j \in W$. Since W contains the algebra $\mathcal{A}$ and is closed with respect to operations $(\cdots)^c$ and $\bigcup\limits_j (\cdots)$, it is a σ-algebra containing $\sigma(\mathcal{A}) = \mathcal{B}$.

Finally, let E be a set in $\mathcal{M}_\mu$ with $\mu(E) = 0$. We can use the definition (1.12) with $\mu^*(E) = \mu(E) = 0$, and $\mu_0 = \mu$ on the algebra $\mathcal{A}$ defined in (I-15). For fixed $\varepsilon > 0$,

$$\exists A_j \in \mathcal{A} \quad \text{such that} \quad E \subseteq \bigcup_{j=1}^{\infty} A_j \quad \text{and} \quad \sum_{j=1}^{\infty} \mu(A_j) < \varepsilon.$$

Similarly to (I-14),

$$\exists \quad \text{open} \quad G_j \supseteq A_j \quad \text{with} \quad \mu(G_j \setminus A_j) < \frac{\varepsilon}{2^j}.$$

Then the open set

$$G := \bigcup_{j=1}^{\infty} G_j \supseteq \bigcup_{j=1}^{\infty} A_j \supseteq E, \quad \text{and} \quad \mu(G) \le \sum_{j=1}^{\infty} \mu(G_j) = \sum_{j=1}^{\infty} \mu(A_j) + \sum_{j=1}^{\infty} \mu(G_j \setminus A_j) < 2\varepsilon.$$

Hence E satisfies (I-12) with $F = \emptyset$ (after redefining ε).

Summarizing the previous arguments, we see that the σ-algebra $W \subseteq \mathcal{M}_\mu$ contains $\mathcal{B} = \sigma(\mathcal{A})$ and all μ-null sets. It remains to note that $\mathcal{M}_\mu$ is the completion of $\mathcal{B}$ with respect to μ, so we must have $W = \mathcal{M}_\mu$. Statement (i) is proved.

(ii). From (I-12) it follows immediately that

$$\mu(E) = \inf \Big\{ \mu(G) : \ E \subseteq \ \text{open } G \Big\} = \sup \Big\{ \mu(F) : \ \text{closed } F \subseteq E \Big\}, \tag{I-17}$$

which includes the first equality in (I-13). In order to prove the second equality, note that

$$\mu(F) = \sup_n \mu(F K_n), \quad \text{where} \quad K_n := [-n, n]^d \subset \mathbb{R}^d.$$

Here FK_n are bounded closed sets in $\mathbb{R}^d$, i.e. compacts, which are contained in $F \subseteq E$. Therefore,

$$\mu(F) \le \sup \Big\{ \mu(K) : \ \text{compact } K \subseteq E \Big\} \le \mu(E).$$

In combination with (I-17), this implies the desired equality. Theorem is proved. $\square$