Appendix 1: Fourier Transforms

Definition 1. The **Fourier transform** of a function $f(x) \in L^1(\mathbb{R}^n)$ is

$$g(\omega) = F[f](\omega) := \int_{\mathbb{R}^n} e^{-i\omega x} f(x)\, dx. \tag{1}$$

Here $x = (x_1, \cdots, x_n)$, $\omega = (\omega_1, \cdots, \omega_n) \in \mathbb{R}^n$,

$$e^{-i\omega x} := \cos\omega x - i \sin\omega x, \qquad \omega x := \omega_1 x_1 + \cdots + \omega_n x_n.$$

Since $|e^{-i\omega x} f| = |f| \in L^1$, by Lebesgue's Dominated Convergence Theorem we have $\lim\limits_{\omega\to\omega_0} g(\omega) = g(\omega_0)$, i.e. $g = F[f]$ is continuous for every $f \in L^1$. Obviously, F is also bounded as an operator from L^1 to L^∞ with $||\, F[f]\, ||_\infty \le ||f||_1$.

First we restrict F to the **Schwartz space** $S \subset C^\infty(\mathbb{R}^n)$ of functions f satisfying

$$\sup_{\mathbb{R}^n} |x^\alpha D^\beta f(x)| = \sup_{\mathbb{R}^n} \left| x^{\alpha_1} \cdots x^{\alpha_n} \frac{\partial^{\beta_1+\cdots+\beta_n} f}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}} \right| < \infty$$

for all multi-indices $\alpha, \beta \ge 0$. Since $(1+|x|^2)^n$ is a polynomial, and $(1+|x|^2)^{-n} \in L^1$, we also have

$$\left|(1+|x|^2)^n x^\alpha D^\beta f\right| \le C(\alpha,\beta) = \text{const} < \infty, \quad x^\alpha D^\beta f \in L^1, \quad \text{and} \quad F[x^\alpha D^\beta f] \in L^\infty.$$

Lemma 1. *For $f \in S$, $\quad g(\omega) := F[f](\omega)$, and all multi-indices $\alpha, \beta \ge 0$, we have:*

$$(\mathbf{a})\ \omega^\alpha g(\omega) = F\big[(-iD)^\alpha f\big](\omega); \qquad (\mathbf{b})\ D^\beta g(\omega) = F\big[(-ix)^\beta f\big](\omega).$$

Proof. The property (a) follows by integration by parts, (b) – by differentiation of the equality (1). □

Corollary 1. *If $f \in S$, then $g := F[f] \in S$.*

Proof. By the previous lemma, $\omega^\alpha D^\beta g$ is a finite linear combination of $F[x^\mu D^\nu f]$ with multi-indices $\mu, \nu \ge 0$. Since all $x^\mu D^\nu f$ belong to L^1, we have $|\omega^\alpha D^\beta g| \le C = \text{const} < \infty$. □

We also define the **inverse Fourier transform** of any function $g(\omega) \in S$ by the formula

$$f(x) = F^{-1}[g](x) := (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i\omega x} g(\omega)\, d\omega = (2\pi)^{-n} \overline{F[\,\overline{g}\,]}. \tag{2}$$

From Corollary 1 and the last equality in (2) it follows that if $g \in S$, then $F^{-1}[g] \in S$. We will show that indeed, F^{-1} is the inverse operator of F on S (equalities (10) in Theorem 2 below). In the following example, we check these equalities for $f = \varphi := e^{-\frac{x^2}{2}}$.

Example 1. We will find the Fourier transform of the function $\varphi(x) = e^{-\frac{x^2}{2}}$ on $\mathbb{R}^1$. Since φ is an even function, we have

$$g(\omega) = F[\varphi](\omega) = \int_{\mathbb{R}^1} \cos\omega x \cdot e^{-\frac{x^2}{2}}\, dx.$$

Using polar coordinates, we get

$$g^2(0) = \int\limits_{\mathbb{R}^1} e^{-\frac{x^2}{2}}\, dx \cdot \int\limits_{\mathbb{R}^1} e^{-\frac{y^2}{2}}\, dy = \iint\limits_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}\, dx\, dy = \int\limits_0^{2\pi} d\theta \int\limits_0^{\infty} e^{-\frac{r^2}{2}} r\, dr = 2\pi,$$

hence $g(0) = \sqrt{2\pi}$. Further,

$$g'(\omega) = -\int\limits_{\mathbb{R}^1} \sin\omega x \cdot x e^{-\frac{x^2}{2}}\, dx = \int\limits_{\mathbb{R}^1} \sin\omega x \cdot d\varphi(x) = -\int\limits_{\mathbb{R}^1} \omega\cos\omega x \cdot \varphi(x)\, dx = -\omega g(\omega),$$

$$(\ln g)' = -\omega, \qquad \ln g = \text{const} - \frac{\omega^2}{2},$$

and since $g(0) = \sqrt{2\pi}$,

$$g(\omega) := F[\varphi](\omega) = \text{const} \cdot e^{-\frac{\omega^2}{2}} = \sqrt{2\pi} \cdot e^{-\frac{\omega^2}{2}}.$$

We will use the same notation φ for the function

$$\varphi(x) = e^{-\frac{x^2}{2}} = e^{-\frac{1}{2}\sum x_k^2} \qquad \text{on} \quad \mathbb{R}^n.$$

Its Fourier transform is represented as the product:

$$g(\omega) = \int\limits_{\mathbb{R}^n} e^{-i\omega x} e^{-\frac{x^2}{2}}\, dx = \prod_{k=1}^{n} \int\limits_{\mathbb{R}^1} e^{-i\omega_k x_k} e^{-\frac{x_k^2}{2}}\, dx_k = \prod_{k=1}^{n} F[\varphi](\omega_k),$$

and by the above formula,

$$F[\varphi](\omega) = F\left[e^{-\frac{x^2}{2}}\right](\omega) = \prod_{k=1}^{n} \left(\sqrt{2\pi} \cdot e^{-\frac{\omega_k^2}{2}}\right) = (2\pi)^{\frac{n}{2}} e^{-\frac{\omega^2}{2}} = (2\pi)^{\frac{n}{2}} \varphi(\omega). \tag{3}$$

The previous calculations remain the same if we replace i by $-i$. Therefore,

$$F^{-1}[\varphi](x) = (2\pi)^{-n} F[\varphi](x) = (2\pi)^{-\frac{n}{2}} \varphi(x),$$

and

$$F[F^{-1}[\varphi]](x) = (2\pi)^{-\frac{n}{2}} F[\varphi](x) = \varphi(x), \qquad F^{-1}[F[\varphi]](x) = (2\pi)^{\frac{n}{2}} F^{-1}[\varphi](x) = \varphi(x). \tag{4}$$

□

Theorem 1. *For any constants $k > 0$ and $h \in \mathbb{R}^n$, operators F and F^{-1} defined by formulas (1) and (2) on S, satisfy the equalities*

$$F[f(kx)](\omega) = k^{-n} F[f(x)](k^{-1}\omega), \qquad F^{-1}[g(k\omega)](x) = k^{-n} F^{-1}[g(\omega)](k^{-1}x), \tag{5}$$

$$F[f(x+h)](\omega) = e^{i\omega h} F[f(x)](\omega), \qquad F^{-1}[g(\omega+h)](x) = e^{-ihx} F^{-1}[g(\omega)](x), \tag{6}$$

$$F[e^{ihx} f(x)](\omega) = F[f(x)](\omega - h), \qquad F^{-1}[e^{ih\omega} g(\omega)](x) = F^{-1}[g(\omega)](x+h). \tag{7}$$

$$F[f * g] = F[f] \cdot F[g]. \tag{8}$$

Proof of (5)–(7) is easy to obtain by changing the variables. The equality (8) follows from Fubini's theorem:

$$F[f * g](\omega) = \int e^{-i\omega x}\left[\int f(x-t)g(t)\,dt\right]dx$$

$$= \int e^{-i\omega t}g(t)\left[\int e^{-i\omega(x-t)}f(x-t)\,dx\right]dt = F[f](\omega)\cdot\int e^{-i\omega t}g(t)\,dt = F[f](\omega)\cdot F[g](\omega).$$

□

Note that the complex-valued functions $f:\ \mathbb{R}^n \to C$ in $L^2(\mathbb{R}^n)$ compose a Hilbert space with the **inner** (or **scalar**) **product**

$$\langle f, g\rangle := \int\limits_{\mathbb{R}^n} f\overline{g}\,dx.$$

Theorem 2. *For all $f, g \in S$, we have*

$$\langle F[f], g\rangle = \int\limits_{\mathbb{R}^n} F[f](\omega)\cdot\overline{g(\omega)}\,d\omega = (2\pi)^n\langle f, F^{-1}[g]\rangle. \tag{9}$$

Moreover,

$$F^{-1}\big[F[f]\big] = f, \qquad F\big[F^{-1}[f]\big] = f, \tag{10}$$

and the following **Plancherel equalities** *hold true:*

$$\|\, F[f]\,\|_2^2 = (2\pi)^n\|\, f\,\|_2^2, \qquad \|\, f\,\|_2^2 = (2\pi)^n\|\, F^{-1}[f]\,\|_2^2. \tag{11}$$

Proof. The equality (9) follows from Fubini's theorem:

$$\langle F[f], g\rangle = \int\left[\int e^{-i\omega x}f(x)\,dx\right]\overline{g(\omega)}\,d\omega = \int f(x)\left[\overline{\int e^{i\omega x}g(\omega)\,d\omega}\right]dx = (2\pi)^n\langle f, F^{-1}[g]\rangle.$$

Further, denote by S_0 the set of all functions $f \in S$ satisfying (10). By Theorem 1 with $g := F[f]$, we have

$$F[f(kx)](\omega) = k^{-n}g(k^{-1}\omega), \qquad F^{-1}[k^{-n}g(k^{-1}\omega)](x) = f(kx);$$
$$F[f(x+h)](\omega) = e^{i\omega h}g(\omega), \qquad F^{-1}[e^{i\omega h}g(\omega)](x) = f(x+h).$$

In other words, if $f \in S_0$, then $f(kx)$ and $f(x+h)$ satisfy the first equality in (10). The second equality follows from the relation $F^{-1}[f] = (2\pi)^{-n}\overline{F[\overline{f}\,]}$. Therefore, from $f \in S_0$ it follows that $f(kx),\ f(x+h) \in S_0$.

By virtue of (4), we know that $\varphi(x) = e^{-\frac{x^2}{2}} \in S_0$. Then

$$K(x) := (2\pi)^{-\frac{n}{2}}\varphi(x), \quad K^\varepsilon(x) := \varepsilon^{-n}K(\varepsilon^{-1}x), \quad \text{and} \quad K^\varepsilon(x-t)$$

belong to S_0 for all $\varepsilon > 0$ and $t \in \mathbb{R}^n$. It is easy to verify that

$$f_\varepsilon(x) := (f * K^\varepsilon)(x) = \int\limits_{\mathbb{R}^n} f(t)\,K^\varepsilon(x-t)\,dt = \int\limits_{\mathbb{R}^n} f(x-\varepsilon y)\,K(y)\,dy$$

belong to S for for all $f \in S$ and $\varepsilon > 0$. In addition,

$$F^{-1}\big[F[f_\varepsilon]\big] = \int\limits_{\mathbb{R}^n} f(t)\, F^{-1}\big[F[K^\varepsilon(x-t)]\big]\, dt = \int\limits_{\mathbb{R}^n} f(t) K^\varepsilon(x-t)\, dt = f_\varepsilon,$$

and similarly, $F\big[F^{-1}[f_\varepsilon]\big] = f_\varepsilon$. This means that in fact we have $f_\varepsilon \in S_0$.

By our choice of constants, we have $\int K(y)\, dy = 1$. Hence

$$(f_\varepsilon - f)(x) = \int\limits_{\mathbb{R}^n} \big[f(x-\varepsilon y) - f(x)\big] \cdot K(y)\, dy.$$

Using Minkowski's integral inequality, we estimate the L^2-norm as follows:

$$||f_\varepsilon - f||_2 \le \int\limits_{\mathbb{R}^n} ||f(x-\varepsilon y) - f(x)||_2 \cdot K(y)\, dy.$$

We have $||f(x-h) - f(x)||_2 \to 0$ as $h \to 0$, even for $f \in L^2$. Then by the Dominated Convergence Theorem, $||f_\varepsilon - f||_2 \to 0$ as $\varepsilon \searrow 0$.

Finally, if $f \in S$ satisfies (10), then (11) follows from (9) with $g := F[f]$. The previous argument shows that these equalities hold true on a family $S_0 \subseteq S$, which is dense in S_0 with respect to the L^2-norm. By standard approximation, these properties are extended to the whole class S, i.e. $S_0 = S$.

□

The Plancherel equalities allow to define Fourier transforms F and F^{-1} for functions $f \in L^2$ as limits in L^2:

$$F[f] := \lim_{n\to\infty} F[f_n], \quad F^{-1}[f] := \lim_{n\to\infty} F^{-1}[f_n], \quad \text{where} \quad f = \lim_{n\to\infty} f_n, \quad f_n \in S.$$

Then "by continuity", all the equalities (5)–(7) and (9)–(11) also hold true for functions in L^2.

Example 2. The Fourier transform of the function $f(x) := e^{-k|x|}$ on $\mathbb{R}^1$, where $k = \text{const} > 0$, is

$$g(\omega) := F[f](\omega) = \int\limits_{\mathbb{R}^1} e^{-i\omega x - k|x|} = 2 \cdot \mathrm{Re} \int\limits_0^\infty e^{-(k+i\omega)x} dx = 2 \cdot \mathrm{Re}\, \frac{1}{k+i\omega} = \frac{2k}{k^2+\omega^2}.$$

Since g is an even function, we also have

$$F[g](\omega) := \int\limits_{\mathbb{R}^1} e^{-i\omega x} g(x)\, dx = \int\limits_{\mathbb{R}^1} e^{i\omega x} g(x)\, dx = 2\pi \cdot F^{-1}[g](\omega) = 2\pi \cdot f(\omega),$$

and

$$F\Big[\frac{k}{k^2+x^2}\Big](\omega) = \frac{1}{2} \cdot F[g](\omega) = \pi \cdot e^{-k|\omega|}.$$

□

Definition 2. For $n = 0, 1, 2, \ldots$, the **Hermite polynomials** are defined as

$$H_n(x) := (-1)^n e^{x^2} \Big(e^{-x^2}\Big)^{(n)},$$

so that $H_0 = 1, H_1 = 2x$, etc. The corresponding **Hermite functions** $\varphi_n(x) := H_n(x)\, e^{-\frac{x^2}{2}}$.

In particular, $\varphi_0(x) = e^{-\frac{x^2}{2}} = \varphi(x)$ in Example 1. All functions φ_n belong to the Schwartz space S. Using integration by parts, it is easy to check that the system $\{\varphi_n\}$ is orthogonal in $L^2 := L^2(\mathbb{R}^1)$:

$$\langle \varphi_m, \varphi_n \rangle := \int\limits_{\mathbb{R}^1} \varphi_m \varphi_n dx = \int\limits_{\mathbb{R}^1} H_m H_n e^{-x^2} dx = 0 \quad \text{for} \quad m \neq n.$$

Theorem 3. *The system of Hermite functions $\{\varphi_n\}$ is complete in L^2, i.e. from $f \in L^2$ and $\langle \varphi_n, f \rangle = 0$ for all n it follows $f = 0$ a.e.*

Proof. The assumption $\langle \varphi_n, f \rangle = 0$ for all n is equivalent to $\langle x^n \varphi, f \rangle = 0$ for all n, because every x^n is a linear combination of H_k, $k \leq n$, and correspondingly, $x^n \varphi$ is a linear combination of $H_k \varphi = \varphi_k$, $k \leq n$. Consider the Fourier transform

$$g(\omega) := F[\varphi f](\omega) = \int\limits_{\mathbb{R}^1} e^{-i\omega x - \frac{x^2}{2}} f(x)\, dx.$$

This integral is well defined for complex $\omega = \omega_1 + i\omega_2$, and $g(\omega)$ is analytic in the whole complex plane C. By our assumptions, all the derivatives

$$g^{(n)}(0) = \int\limits_{\mathbb{R}^1} (-ix)^n e^{-\frac{x^2}{2}} f(x)\, dx = (-i)^n \langle x^n \varphi, f \rangle = 0.$$

By uniqueness for analytic functions, we must have $g \equiv 0$. Finally, from the Plancherel equality it follows

$$2\pi \cdot ||\varphi f||_2^2 = ||g||_2^2 = 0,$$

so that $f = 0$ a.e. □

Theorem 4. *The Hermite functions φ_n are eigenfunctions of the Fourier transform:*

$$F[\varphi_n] = c_n \varphi_n, \quad \text{where} \quad c_n := (-i)^n \sqrt{2\pi} \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{12}$$

Proof. We know that this property holds true for $n = 0$ with $c_0 := \sqrt{2\pi}$. Moreover,

$$(x - D)\varphi_k = (-1)^k (x - D) \left[e^{\frac{x^2}{2}} \left(e^{-x^2} \right)^{(k)} \right] = (-1)^{k+1} e^{\frac{x^2}{2}} \left(e^{-x^2} \right)^{(k+1)} = \varphi_{k+1},$$

so that by induction, $\varphi_n = (x - D)^n \varphi$ for all $n = 0, 1, 2, \ldots$. Note that by Lemma 1,

$$F\big[(x - D) f\big] = -i\, F\big[(-i)(D - x) f\,\big] = -i\, (\omega - D)\, F[f] \quad \text{for} \quad f \in S.$$

Therefore,

$$F[\varphi_n] = F\big[(x - D)^n \varphi\big] = (-i)^n (\omega - D)^n F[\varphi] = (-i)^n \sqrt{2\pi} \cdot (\omega - D)^n \varphi = (-i)^n \sqrt{2\pi} \cdot \varphi_n.$$

Theorem is proved. □

At the conclusion, we prove a few relations between the Fourier operator F, the differential operator $L := D^2 - x^2$, and the Hermite functions $\varphi_n := H_n \varphi$.

Theorem 5. (a). *The Fourier operator F is commutative with $L := D^2 - x^2$ on S:*

$$F[Lf] = LF[f] \quad \text{for} \quad f \in S. \tag{13}$$

In particular, if $Lf := f'' - x^2 f = 0$, then $g(\omega) := F[f](\omega)$ also satisfies $Lg := g'' - \omega^2 g = 0$.

(b). *The Hermite functions $\varphi_n := H_n \varphi$ are eigenfunctions of L:*

$$L\varphi_n := \varphi_n'' - x^2 \varphi_n = \lambda_n \varphi_n \quad \text{with} \quad \lambda_n := -(2n+1) \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{14}$$

(c). *The Hermite polynomials H_n satisfy the* **Hermite equation**

$$y'' - 2xy' = \mu y \quad \text{with} \quad \mu = 2n \quad \text{for} \quad n = 0, 1, 2, \ldots. \tag{15}$$

Proof. (a). By Lemma 1, the functions f and $g := F[f]$ in S satisfy

$$F[D^2 f] = -F[(-iD)^2 f] = -\omega^2 g, \qquad F[-x^2 f] = F[(-ix)^2 f] = D^2 g.$$

From these relations, the equality (13) follows:

$$F[Lf] = F[D^2 f - x^2 f] = -\omega^2 g + D^2 g = Lg = LF[f].$$

(b) and **(c)**. We will try to find polynomials P_n of degree n (eventually $P_n = \text{const} \cdot H_n$) such that $\psi_n := P_n \varphi$ satisfy $L\psi_n = \psi_n'' - x^2 \psi_n = \lambda \psi_n$ with a constant λ (depending on n). Since $\varphi(x) := e^{-\frac{x^2}{2}}$ satisfies $\varphi' = -x\varphi$, $\varphi'' = (x^2 - 1)\varphi$, we get

$$L\psi_n = P_n''\varphi + 2P_n'\varphi' + P_n\varphi'' - x^2 P_n \varphi = (P_n'' - 2xP_n' - P_n)\varphi = \lambda P_n \varphi.$$

Here $P_n = \sum\limits_{k=0}^{n} a_k x^k$, $a_n \neq 0$. Comparing the coefficients of x^n in both sides, we see that the equality is only possible if $\lambda = \lambda_n := -(2n+1)$. One can select $a_n \neq 0$ in an arbitrary way, and then the remaining coefficients a_k are uniquely defined by a standard recurrent procedure.

From the equalities $L\psi_k = \lambda_k \psi_k$ it follows

$$(\psi_m' \psi_n - \psi_n' \psi_m)' = \psi_m'' \psi_n - \psi_n'' \psi_m = (\lambda_m - \lambda_n)\psi_m \psi_n.$$

Integrating over $\mathbb{R}^1$ yields $0 = (\lambda_m - \lambda_n) \cdot \langle \psi_m, \psi_n \rangle$, so that $\{\psi_n\}$ is an orthogonal system in L^2. Note that both $\{\varphi_n := H_n\varphi\}$ and $\{\psi_n := P_n\varphi\}$ can be obtained by orthogonalization of $\{x^n\varphi\}$, i.e. $\langle \varphi_n, x^k\varphi \rangle = \langle \psi_n, x^k \varphi \rangle = 0$ for all $k \leq n-1$. From this observation it follows that $\varphi_n = \text{const} \cdot \psi_n$ and $H_n = \text{const} \cdot P_n$. Finally, since $L\psi_n = \lambda_n \psi_n$ and $P_n'' - 2xP_n' + 2nP_n = 0$, the functions $\varphi_n := H_n\varphi$ satisfy (14), and $y = H_n$ satisfy (15). Theorem is proved.

□