Math 8602: REAL ANALYSIS. Spring 2016.
Final Exam. Problems and Solutions.

#1. Let f be a Lebesgue measurable function on $\mathbb{R}^1$ such that

$$f(x+y) = f(x) + f(y) \quad \text{for all} \quad x, y \in \mathbb{R}^1.$$

Show that $f(x) = cx$ for some constant c.

Proof. We rely on the definition of a Lebesgue measurable function $f: \mathbb{R}^1 \to \mathbb{R}^1$ on p. 44, which does not assume values $\pm\infty$. The statement is easily extended to functions $f = f_1 + if_2 : \mathbb{R}^1 \to \mathbb{C}$, where f_1 and f_2 are real-valued functions. Indeed, from $f(x+y) = f(x) + f(y)$ it follows $f_{1,2}(x+y) = f_{1,2}(x) + f_{1,2}(y)$. If the statement holds true for $f_{1,2}$, then $f_{1,2}(x) = c_{1,2}x$, and $f(x) = cx$ with $c := c_1 + ic_2$.

Note that the statement in Homework 5, Problem 2, was proved under the assumption $|f| < \infty$ a.e. It can be applied to both functions f and $-f$, because

$$f\Big(\frac{x+y}{2}\Big) = \frac{1}{2} \cdot \Big[f\Big(\frac{x+y}{2}\Big) + f\Big(\frac{x+y}{2}\Big)\Big] = \frac{f(x+y)}{2} = \frac{f(x)+f(y)}{2}.$$

Therefore, both functions f and $-f$ are convex, which is only possible if $f(x)$ is a linear function. Finally, $f(0) = f(0+0) = f(0) + f(0)$, so that $f(0) = 0$, and $f(x) = cx$ for some constant c.

#2. Let f, g be function in the linear space $L^p(X, \mathcal{M}, \mu)$, $0 < p < \infty$, with quasinorm

$$||f||_p := \Big(\int |f|^p d\mu\Big)^{1/p}.$$

Show that

$$||f+g||_p \le K(p) \cdot \big(||f||_p + ||g||_p\big),$$

where $K(p)$ is a constant such that $K(p) \searrow 1$ as $p \nearrow 1$.

Proof. It is known that $||\cdot||_p$ is a norm (i.e. $K(p) = 1$) if $p \ge 1$, so that it suffices to consider the case $0 < p < 1$. Note that

$$|f+g|^p \le |f|^p + |g|^p \quad \text{for} \quad 0 < p < 1. \tag{1}$$

Indeed, one can assume that $f > 0$, $g > 0$. Then (1) is equivalent to

$$1 \le F(t) := t^p + (1-t)^p, \quad \text{where} \quad t := \frac{f}{f+g} \in (0,1).$$

Since $F''(t) = p(p-1) \cdot \big[t^{p-2} + (1-t)^{p-2}\big] < 0$ on $(0,1)$, the function $F(t)$ is concave on $[0,1]$. In addition, $F(0) = F(1) = 1$, and the inequality $F(t) \ge 1$ follows.

Using (1), one can write

$$\int |f+g|^p d\mu \le A_1 + A_2, \quad \text{where} \quad A_1 := \int |f|^p d\mu, \quad A_2 := \int |g|^p d\mu.$$

Now Hölder's inequality with $p_1 := 1/p > 1$, $1/q_1 = 1 - 1/p_1 = 1 - p$, implies

$$A_1 + A_2 = \sum_{j=1}^{2} 1 \cdot A_j \le 2^{1/q_1} \cdot \Big(\sum_{j=1}^{2} A_j^{p_1}\Big)^{1/p_1} = 2^{1/q_1} \cdot \big(||f||_p + ||g||_p\big)^p,$$

which in turn yields the desired inequality

$$||f+g||_p \le (A_1 + A_2)^{1/p} \le K(p) \cdot \big(||f||_p + ||g||_p\big), \quad \text{where} \quad K(p) := 2^{\frac{1-p}{p}} = 2^{\frac{1}{p}-1} \searrow 1 \quad \text{as} \quad p \nearrow 1.$$

#3. Let A and B be **disjoint convex compact** sets in $\mathbb{R}^n$.
(a). Show that there exist $a \in A$ and $b \in B$ such that

$$|a-b| = \min\{|x-y| : \ x \in A, \ y \in B\}.$$

(b). Use this fact to prove that there is a linear function $l(x) := c_0 + (c, x)$ with constants $c_0 \in \mathbb{R}^1$ and $c \in \mathbb{R}^n$, such that $l(x) < 0$ on A, and $l(x) > 0$ on B.

Proof. (a). Since A and B are bounded, there are sequences $\{x_j\} \subseteq A$ and $\{y_j\} \subseteq B$, such that

$$|x_j - y_j| \ \to \ d(A,B) := \inf\{|x-y| : \ x \in A, \ y \in B\} \quad \text{as} \quad j \to \infty.$$

By compactness of A, there is a convergent subsequence $a_k := x_{j_k} \to a \in A$ as $k \to \infty$. Then

$$\lim_{k\to\infty} |a - b_k| = \lim_{k\to\infty} |a_k - b_k| = d(A,B), \quad \text{where} \quad b_k := y_{j_k}.$$

By compactness of B, there is a convergent subsequence $b_{k_l} \to b \in B$ as $l \to \infty$. Finally,

$$|a-b| = \lim_{l\to\infty} |a - b_{k_l}| = d(A,B).$$

One can write "min" in place of "inf" in the definition of $d(A,B)$, because it is attained at the points $x = a$, $y = b$.

(b). We define

$$l(x) := (c, x - x_0), \quad \text{where} \quad c := b - a \neq 0, \quad x_0 := \frac{1}{2} \cdot (b+a).$$

Of course, one can rewrite it as $l(x) = c_0 + (c,x)$ with $c_0 := -(c, x_0)$. It is easy to see that

$$l(a) = -\frac{1}{2} \cdot |b-a|^2 < 0 < l(b) = \frac{1}{2} \cdot |b-a|^2.$$

We claim that

$$l(x) \le l(a) < 0 \quad \text{on} \quad A, \qquad l(x) \ge l(b) < 0 \quad \text{on} \quad B. \tag{2}$$

This means that A and B are separated in $\mathbb{R}^n$ by the strip $\{x \in \mathbb{R}^n : \ l(a) < l(x) < l(b)\}$.

We will derive (2) by a contradiction argument. Indeed, suppose that $l(p) > l(a)$ at some point $p \in A$. By convexity of A,

$$a + \varepsilon v \in A \quad \text{for every} \quad \varepsilon \in [0,1], \quad \text{where} \quad v := p - a \neq 0.$$

Note that $(v,c) = (p,c) - (a,c) = l(p) - l(a) > 0$. Since $a + \varepsilon v \in A$ and $b \in B$, we get for small $\varepsilon > 0$:

$$|c|^2 = |a-b|^2 = d^2(A,B) \le |a + \varepsilon v - b|^2 = |\varepsilon v - c|^2 = \varepsilon^2 |v|^2 - 2\varepsilon(v,c) + |c|^2 < |c|^2.$$

This contradiction proves the first assertion in (2). The second one can be proved quite similarly.

#4. For $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$ with $p \ge 1$, show that the convolution

$$(f * g)(x) := \int\limits_{\mathbb{R}^n} f(x-y) g(y)\, dy$$

belongs to $L^p(\mathbb{R}^n)$ and satisfies $||f * g||_p \le ||f||_1 \cdot ||g||_p$.

Proof. This property is obvious in the case $p = \infty$, so we assume that $1 \le p < \infty$. Without loss of generality, we can also assume that $f \ge 0$, $g \ge 0$, and moreover, f and g are bounded functions vanishing outside a compact set in $\mathbb{R}^n$. The general case follows easily by the monotone convergence theorem.

Let q be a conjugate exponent to p, i.e. $p^{-1}+q^{-1}=1$. By the Riesz representation in Proposition 6.13, we have

$$||f*g||_p=\sup_{||h||_q\le 1}\int (f*g)h\,dx.$$

Here by the Fubini-Tonelli theorem and Hölder's inequality,

$$\int (f*g)h\,dx=\int f(y)\Big[\int g(x-y)h(x)\,dx\Big]\,dy\le\int f(y)\cdot||g||_p\cdot||h||_q\,dy=||f||_1\cdot||g||_p\cdot||h||_q.$$

Therefore, $||f*g||_p\le||f||_1\cdot||g||_p$.

#5. Consider the Dirichlet kernel

$$K_\varepsilon(x):=\frac{\sin(\varepsilon^{-1}x)}{\pi x},\qquad x\in\mathbb{R}^1,\quad \varepsilon>0.$$

(a). Show that for every continuous function f with compact support in $\mathbb{R}^1$, we have

$$\int\limits_{\mathbb{R}^1}|f_\varepsilon|^2dx\le C\cdot\int\limits_{\mathbb{R}^1}|f|^2dx,\quad\text{where}\quad f_\varepsilon:=K_\varepsilon*f,$$

with a positive constant C. Find the smallest possible C.

(b). Let

$$f(x)=\sum_{k=1}^{\infty}\frac{\sin(2^{k^3}x)}{k^2},\quad 0\le x\le\pi;\quad\text{and}\quad f\equiv 0\quad\text{on}\quad\mathbb{R}^1\setminus[0,\pi].$$

Show that

$$\lim_{\varepsilon\searrow 0}|f_\varepsilon(0)|=\infty.$$

Proof. (a). Note that

$$K_\varepsilon(\omega)=\frac{\sin(\varepsilon^{-1}\omega)}{\pi\omega}=\frac{1}{2\pi}\int\limits_{-1/\varepsilon}^{1/\varepsilon}\cos\omega x\,dx=\frac{1}{2\pi}\cdot F[I_\varepsilon],$$

where I_ε is the indicator of $[-1/\varepsilon,1/\varepsilon]$. Comparing formulas for the direct and inverse Fourier transforms (formulas (1) and (2) in Appendix 1), we get

$$F[K_\varepsilon]=2\pi\cdot F^{-1}[K_\varepsilon]=I_\varepsilon$$

$$\Longrightarrow\quad|\,F[f_\varepsilon]\,|=|\,F[K_\varepsilon*f]\,|=|\,F[K_\varepsilon]\cdot F[f]\,|\le|\,F[f]\,|\quad\Longrightarrow\quad||\,F[f_\varepsilon]\,||_2\le||\,F[f]\,||.$$

By the Plancherel Theorem 8.29 (or equality (11) in Appendix 1), we get $||f_\varepsilon||_2\le||f||_2$, i.e. one can take $C=1$. This is a minimal possible constant, because by the dominated convergence theorem,

$$2\pi\cdot\lim_{\varepsilon\searrow 0}\int|f_\varepsilon|^2dx=\lim_{\varepsilon\searrow 0}\int|F[f_\varepsilon]\,|^2d\omega=\lim_{\varepsilon\searrow 0}\int|I_\varepsilon\cdot F[f]\,|^2d\omega=\int|F[f]\,|^2d\omega=2\pi\cdot\int|f|^2dx.$$

(b). We take $\varepsilon=\varepsilon_n=2^{-n^3}$. Then

$$f_\varepsilon(0)=\int\limits_0^\pi f(t)K_\varepsilon(-t)\,dt=\sum_{k=1}^{\infty}I_k,\quad\text{where}\quad I_k=\frac{1}{k^2}\int\limits_0^\pi\frac{\sin(2^{n^3}t)\,\sin(2^{k^3}t)}{\pi t}\,dt.$$

Note that for $A \geq 2B > 0$,

$$\int_0^\pi \frac{\sin At \cdot \sin Bt}{t}\, dt = \int_0^\pi \frac{1-\cos(A+B)t}{2t}\, dt - \int_0^\pi \frac{1-\cos(A-B)t}{2t}\, dt$$

$$= \left(\int_0^{(A+B)\pi} - \int_0^{(A-B)\pi} \right) \frac{1-\cos s}{2s}\, ds = \int_{(A-B)\pi}^{(A+B)\pi} \frac{1-\cos s}{2s}\, ds.$$

It follows

$$0 \leq \int_0^\pi \frac{\sin At \cdot \sin Bt}{t}\, dt \leq \int_{(A-B)\pi}^{(A+B)\pi} \frac{ds}{s} = \ln \frac{A+B}{A-B} \leq \ln 3.$$

Therefore,

$$0 \leq \sum_{k \neq n} I_k \leq \sum_{k \neq n} \frac{\ln 3}{k^2 \pi} < \sum_{k=1}^{\infty} \frac{\ln 3}{k^2 \pi} =: C_0 = \text{const} < \infty.$$

On the other hand, for $A = B = 2^{n^3}$,

$$\int_0^\pi \frac{\sin^2 At}{t}\, dt = \int_0^{2A\pi} \frac{1-\cos s}{2s}\, ds \geq \int_{2\pi}^{2A\pi} \frac{1-\cos s}{2s}\, ds = \frac{\ln A}{2} - \int_{2\pi}^{2A\pi} \frac{d(\sin s)}{2s} = \frac{\ln A}{2} - \int_{2\pi}^{2A\pi} \frac{\sin s\, ds}{2s^2}$$

$$> \frac{\ln A}{2} - \int_1^\infty \frac{ds}{2s^2} = \frac{\ln A - 1}{2} = \frac{n^3 \ln 2 - 1}{2}.$$

Hence

$$I_n := \frac{1}{n^2} \int_0^\pi \frac{\sin^2(2^{n^3} t)}{\pi t}\, dt = \frac{n^3 \ln 2 - 1}{2\pi n^2} \to \infty \quad \text{as} \quad n \to \infty.$$

This implies

$$f_{\varepsilon_n}(0) = I_n + \sum_{k \neq n} I_k \to \infty \quad \text{as} \quad n \to \infty.$$

#6. Consider the family of functions on $\mathbb{R}^1$:

$$K_\varepsilon(x) := \frac{\varepsilon}{\pi(x^2 + \varepsilon^2)}, \quad \varepsilon > 0.$$

Show that the convolution

$$K_{\varepsilon_1} * K_{\varepsilon_2} \equiv K_{\varepsilon_1 + \varepsilon_2} \quad \text{for} \quad \varepsilon_1, \varepsilon_2 > 0.$$

Proof. By Example 2 in Appendix 1, the Fourier transform $F[K_\varepsilon](\omega) = e^{-\varepsilon|\omega|}$. Therefore,

$$\begin{aligned} F[K_{\varepsilon_1} * K_{\varepsilon_2}](\omega) &= F[K_{\varepsilon_1}](\omega) \cdot F[K_{\varepsilon_2}](\omega) \\ &= e^{-\varepsilon_1|\omega|} \cdot e^{-\varepsilon_2|\omega|} = e^{-(\varepsilon_1+\varepsilon_2)|\omega|} = F[K_{\varepsilon_1+\varepsilon_2}](\omega) \quad \text{for} \quad \varepsilon_1,\, \varepsilon_2 > 0. \end{aligned}$$

Since F is invertible in L^2, this implies the desired identity.