Math 8602: REAL ANALYSIS. Spring 2016
Some problems for Midterm Exam #2 on Wednesday, April 6.

You will have 50 minutes (10:10 am–11:00 am) to work on 4 problems, 2 of which will be selected from the following list.

No books and electronic devices. You can use class notes, including Appendices and Solutions to Homeworks.

#1. Show that for any two Borel measurable sets $E_1, E_2 \subset \mathbb{R}^1$ with finite Borel measure, the convolution (see Sec. 8.2, p.239 in the textbook)

$$f(x) = \big(I_{E_1} * I_{E_2}\big)(x), \quad \text{where} \quad I_E(y) = \begin{cases} 1 & \text{if} \quad y \in E, \\ 0 & \text{if} \quad y \notin E, \end{cases}$$

is continuous on $\mathbb{R}^1$.

#2. Show that the function

$$f(x) = \int\limits_{\mathbb{R}} \frac{\ln|x-y|}{|x-y|^{1/2}(1+y^2)}\, dy$$

is finite a.e. with respect to the Lebesgue measure.

#3. Let $f, f_1, f_2, \cdots$ be real measurable functions on $\mathbb{R}$, such that $f_n \to f$ almost everywhere as $n \to \infty$,

$$\int\limits_{\mathbb{R}} f(x)\, dx = 1, \quad \int\limits_{\mathbb{R}} f_n(x)\, dx = 1, \quad \text{and} \quad f_n \ge 0 \quad \text{for all } n.$$

(a) Show that $f_n \to f$ in $L^1(R)$ as $n \to \infty$.

(b) Show that (a) may fail if the assumption $\int\limits_{\mathbb{R}} f(x)\, dx = 1$ is dropped.

(c) Show that (a) may fail if the assumption $f_n \ge 0$ is dropped.

Hint. Consider the functions $g_n = \min\{f, f_n\}$.

#4. Let $f(x) \in L^1_{loc}(\mathbb{R})$ and

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} \qquad \text{for all} \quad x, y \in \mathbb{R}.$$

Show that f is convex on $\mathbb{R}$.

#5. Let μ_1, μ_2 and ν be measures on the same σ-algebra Σ of subsets of E, such that all $\mu_1(E)$, $\mu_2(E)$, $\nu(E) > 0$. Suppose $\mu_1 \ll \nu$, $\mu_2 \perp \nu$, i.e μ_1 is absolutely continuous with respect to ν, and μ_2 is singular with respect to ν. For each of 3 statements (a) $\mu_1 \ll \mu_2$; (b) $\mu_2 \ll \mu_1$; (c) $\mu_1 \perp \mu_2$, prove one of three: (1) always true; (2) always false; (3) true for some μ_1, μ_2 and ν satisfying the above conditions, and false for others.