Math 8602. February 24, 2016. Midterm Exam 1. Problems and Solutions.

Problem 1. Let $f, f_1, f_2, \cdots$ be Lebesque integrable functions on $\mathbb{R}^n$, such that

$$\int |f_k - f| \to 0 \qquad \text{as} \quad k \to \infty. \tag{1}$$

Show that

(a)

$$\sup_k \int |f_k| \le C = \text{const} < \infty;$$

(b)

$$\sup_k \int\limits_{\{|f_k| \ge N\}} |f_k| \to 0 \qquad \text{as} \quad N \to \infty.$$

Proof. 1(a). From (1) it follows that for every $\varepsilon > 0$, there exists a constant K_ε such that

$$\int |f_k - f| \le \varepsilon \quad \text{for every} \quad k \ge K_\varepsilon.$$

In particular, using this property with $\varepsilon = 1$, we conclude that the sequence $\int |f_k - f|$ is bounded, therefore,

$$\sup_k \int |f_k| \le \int |f| + \int |f_k - f| \le C = \text{const} < \infty.$$

1(b). Further, for each $k = 1, 2, \ldots$, the Lebesgue measure of the set $E_{k,N} := \{|f_k| \ge N\}$,

$$m(E_{k,N}) = \int\limits_{E_{k,N}} 1 \le \frac{1}{N} \cdot \int\limits_{E_{k,N}} |f_k| \le \frac{C}{N}.$$

By absolute continuity of the Lebesque integral,

$$\sup_k \int\limits_{E_{k,N}} |f| \to 0 \quad \text{as} \quad N \to \infty.$$

and for each $k = 1, 2, \ldots$,

$$\int\limits_{E_{k,N}} |f_k| \to 0 \quad \text{as} \quad N \to \infty.$$

Therefore, for each $\varepsilon > 0$,

$$\limsup_{N \to \infty} \sup_k \int\limits_{E_{k,N}} |f_k| \le \limsup_{N \to \infty} \sup_{k \ge K_\varepsilon} \left(\int |f_k - f| + \int\limits_{E_{k,N}} |f| \right) \le \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the property (b) follows.

Problem 2. Let $f, f_1, f_2, \cdots$ be Lebesque integrable functions on a unit ball $B \subset \mathbb{R}^n$, such that $f_k \to f$ a.e. as $k \to \infty$. In the previous problem, where all the integrals are taken over B, show that from (a) and (b) it follows (1). Verify whether or not this is true with $\mathbb{R}^n$ in place of B.

Proof. For fixed $N \geq 1$,

$$f_k^{(N)} := \min\big\{|f_k|,\ N\big\} \ \to\ f^{(N)} := \min\big\{|f|,\ N\big\} \quad \text{a.e. in} \quad B \quad \text{as} \quad k \to \infty.$$

By the dominated convergence theorem,

$$\int |f_k^{(N)} - f^{(N)}| \ \to\ 0 \quad \text{as} \quad k \to \infty, \quad \text{and} \quad \int |f^{(N)} - f| \ \to\ 0 \quad \text{as} \quad N \to \infty.$$

Note that we always have

$$|f_k - f| \leq |f_k^{(N)} - f^{(N)}| + |f^{(N)} - f| + |f_k^{(N)} - f_k|.$$

From (b) it follows that

$$\sup_k \int |f_k - f| = \sup_k \int\limits_{E_{k,N}} |f_k - N| \leq \sup_k \int\limits_{E_{k,N}} |f_k| \ \to\ 0 \quad \text{as} \quad N \to \infty.$$

Therefore,

$$\limsup_{k\to\infty} \int |f_k - f| \leq \int |f^{(N)} - f| + \sup_k \int\limits_{E_{k,N}} |f_k| \ \to\ 0 \quad \text{as} \quad N \to \infty.$$

This brings us to (1). Note that the property (a) was not used in the proof. In fact, it follows automatically from (b), because

$$|f_k \leq |f_k| \cdot I_{E_{k,N}} + N.$$

For $\mathbb{R}^n$ in place of B, the properties (a) and (b) do not imply (1): in the case $n = 1$,

$$f_k := \frac{1}{k} \cdot I_{(0,k)} \ \to\ f \equiv 0 \quad \text{as} \quad k \to \infty, \quad \text{with} \quad \int |f_k - f| = 1 \quad \text{for all} \quad k.$$

Problem 3. Let F be a real-valued absolutely continuous function on $[0,1]$ and let its derivative $F' = 0$ a.e. on a set $E \subseteq [0,1]$. Show that the Lebesgue measure $m(F(E)) = 0$.

Proof. Since F is absolutely continuous on $[0,1]$, by Theorem 3.35, there exists $f := F' \in L^1([0,1])$ a.e. By regularity of the Borel measure $d\nu := |f|\,dm$ (Theorem 1.18 in the textbook, or Theorem I-6 in lecture notes), for an arbitrary $\varepsilon > 0$ there is an open set $G \supset E$ such that $\nu(G) < \nu(E) + \varepsilon$. Here we assume that f is extended as $f \equiv 0$ on $\mathbb{R}^1 \setminus [0,1]$.

Since $f = 0$ a.e. on E, we have $\nu(E)$=0, so that $\nu(G) < \varepsilon$. Moreover, an open set G is represented as at most countable union of open intervals I_j. Therefore,

$$m\big(F(E)\big) \subseteq m\big(F(G)\big) \leq \sum_j m\big(F(I_j)\big) \leq \sum_j \int\limits_{I_j} |f|\,dx = \int\limits_G |f|\,dx = \nu(G) < \varepsilon,$$

and since $\varepsilon > 0$ is arbitrary, we must have $m(F(E)) = 0$.

Problem 4. Let $(X, \mathcal{T})$ be a topological space, and let A be dense in X, i.e. $\overline{A} = X$. Then for any open set U, we have $\overline{U} = \overline{U \cap A}$.

Proof. Note that the set $V := U \backslash \overline{U \cap A}$ is open, and $V \cap A = \emptyset$. Then also $V = V \cap X = V \cap \overline{A} = \emptyset$, which means $U \subseteq \overline{U \cap A}$, hence $\overline{U} \subseteq \overline{U \cap A}$. The opposite inclusion is trivial, because $U \cap A \subseteq U$.