Math 8602: REAL ANALYSIS. **Spring 2016**

Homework #1 (due on Wednesday, February 3).
40 points are divided between 4 problems, 10 points each.

#1. Let C be a collection of open balls in $\mathbb{R}^n$. Show that there exists a finite or countable subset $C_1 \subseteq C$ such that

$$\bigcup_{B \in C_1} B = \bigcup_{B \in C} B.$$

#2. By definition on p. 95, a measurable function $f: \mathbb{R}^n \to \mathbb{R}$ is **locally integrable** ($f \in L^1_{loc}$) if

$$\int_K |f(x)|\,dx < \infty \qquad \forall \quad \text{bounded measurable} \quad K \subset \mathbb{R}^n.$$

Show that this definition is equivalent to the following:

$$\forall x \in \mathbb{R}^n, \quad \exists r > 0 \quad \text{such that} \quad \int_{B_r(x)} |f(x)|\,dx < \infty.$$

#3. Let $d\nu = d\lambda + f\,dm$ be the Lebesgue-Radon-Nikodym decomposition of a finite real signed measure on $\mathbb{R}^n$. Show that for the total variations (defined on p. 87) we also have

$$d|\nu| = d|\lambda| + |f|\,dm.$$

#4. For each $x \in [0,1]$, let

$$x = \sum_{k=1}^{\infty} \frac{x_k}{2^k},$$

where $x_k = 0$ or 1, so that x_k are functions of x with values 0 and 1. Show that

$$S_n(x) = \frac{x_1 + x_2 + \cdots + x_n}{n} \to \frac{1}{2} \qquad \text{as} \quad n \to \infty \quad \text{in measure on} \quad [0,1].$$

Hint. First check that the functions

$$f_k(x) = x_k(x) - \frac{1}{2}$$

satisfy

$$\int_0^1 f_j(x) f_k(x)\,dx = 0 \quad \text{for} \quad j \neq k.$$