Math 8602: REAL ANALYSIS. Spring 2016

Homework #5 (**corrected**, due on Wednesday, April 20).
40 points are divided between 4 problems, 10 points each.

#1. Let f be a function in $L^1(\mathbb{R}^1)$. Show that

$$\int_{\mathbb{R}^1} f(x)\sin(\omega x)\,dx \to 0 \quad \text{as} \quad \omega \to \infty.$$

#2. Let $f(x) \in L^1_{loc}(\mathbb{R})$ and

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x)+f(y)}{2} \qquad \text{for all} \quad x, y \in \mathbb{R}.$$

Show that f is convex on $\mathbb{R}$.

#3. Show that

$$H_n(x) := (-1)^n e^{x^2}\Big(e^{-x^2}\Big)^{(n)}$$

are polynomials of degree n (the *Hermite* polynomials) satisfying

$$\int_{-\infty}^{\infty} e^{-x^2} H_k H_n\,dx = 0 \quad \text{for} \quad k \ne n.$$

Derive the equality

$$F(t,x) := \sum_{n=0}^{\infty} \frac{t^n}{n!}\cdot H_n(x) = e^{2tx-t^2}.$$

Remark. One can show that $y = H_n$ satisfies the *Hermite* equation $y'' - 2xy' + 2ny = 0$. In a similar way, the *Laguerre* polynomials

$$L_n(x) := \frac{e^x}{n!}\cdot\Big(e^{-x}x^n\Big)^{(n)} \quad \text{satisfy} \quad \int_0^{\infty} e^{-x} L_k L_n\,dx = 0 \quad \text{for} \quad k \ne n,$$

and $y = L_n$ satisfies the *Laguerre* equation $xy'' + (1-x)y' + ny = 0$.

#4. Let $\{x_n\}$ be a sequence in a Hilbert space $\mathcal{H}$ such that $||x_n|| \le 1$ for all n, and for each $y \in \mathcal{H}$, we have $(x_n, y) \to 0$ as $n \to \infty$. Show that there is a subsequence $\{x_{n_j}\}$ such that

$$\frac{1}{k}\big(x_{n_1} + \cdots + x_{n_k}\big) \to 0 \quad \text{as} \quad k \to \infty.$$