Math 8602: REAL ANALYSIS. Spring 2016

Homework #2. Problems and Solutions.

#1. Let a functions $f \in BV([a,b])$ for every subinterval $[a,b] \subset (0,1)$, and its variation on $[a,b]$ does not exceed a constant $C_0 < \infty$ which does not depend on a, b. Show that there exists

$$\lim_{a \searrow 0} f(a).$$

Proof. Suppose that this limit does not exist. Then

$$M := \limsup_{a \searrow 0} f(a) > m := \liminf_{a \searrow 0} f(a).$$

Fix m_0 and M_0 satisfying $m < m_0 < M_0 < M$. Then there are sequences $a_j, b_j \in (0,1)$ such that

$$1 > b_1 > a_1 > b_2 > a_2 > \cdots > b_k > a_k > \cdots, \quad \text{and} \quad f(b_j) > M_0, \; f(a_j) < m_0 \quad \text{for all} \quad j.$$

The total variation of f on $[a_k, b_1]$,

$$T_f(b_1) - T_f(a_k) \ge \sum_{j=1}^{k} \left| f(b_j) - f(a_j) \right| \ge k \cdot (M_0 - m_0) \to \infty \quad \text{as} \quad k \to \infty,$$

in contradiction to our assumption. Therefore, we must have $M = m$, which means that the corresponding limit exists.

#2. Show that for all $\alpha > 1$ the functions

$$f_\alpha(x) := \sum_{k=1}^{\infty} \frac{\sin\left(2^k x\right)}{2^{k\alpha}} \in BV([0,\pi]),$$

i.e. they have bounded variation on $[0,\pi]$.

Proof. The variation of f_α on $[0,\pi]$,

$$V_0^\pi\left[f_\alpha\right] \le \sum_{k=1}^{\infty} V_0^\pi\left[2^{-k\alpha} \sin\left(2^k \alpha\right)\right] \le \sum_{k=1}^{\infty} \int_0^\pi 2^{-k\alpha} \left| \frac{d}{dx} \sin\left(2^k x\right) \right| dx$$

$$\le \sum_{k=1}^{\infty} \int_0^\pi 2^{k(1-\alpha)} dx = \pi \sum_{k=1}^{\infty} 2^{k(1-\alpha)} < \infty$$

for $\alpha > 1$, i.e. $f_\alpha \in BV([0,\pi])$.

#3. Let constants $\alpha, \beta \in (0,1)$ with $\alpha + \beta > 1$, and let functions f, g satisfy

$$[f]_\alpha := \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty, \quad [g]_\beta := \sup_{x \ne y} \frac{|g(x) - g(y)|}{|x-y|^\beta} < \infty.$$

Show that there exists

$$\lim_{n\to\infty} S_n, \quad \text{where} \quad S_n := \sum_{j=1}^{2^n} f(2^{-n}j)\big[g(2^{-n}j) - g(2^{-n}(j-1))\big].$$

Proof. We rewrite S_n in the form

$$S_n = \sum_{j=1}^{2^n} A_j B_j, \quad \text{where} \quad A_j := f(2^{-n}j), \quad B_j := g(2^{-n}j) - g(2^{-n}(j-1)).$$

Compare this sum with

$$S_{n+1} = \sum_{j=1}^{2^{n+1}} a_j b_j = \sum_{j=1}^{2^n} (a_{2j-1}b_{2j-1} + a_{2j}b_{2j}).$$

The expressions for a, b are similar to those for A, B, with $n+1$ in place of n. Note that $A_j = a_{2j}$ and $B_j = b_{2j-1} + b_{2j}$. Therefore,

$$S_{n+1} - S_n = \sum_{j=1}^{2^n} (a_{2j-1} - a_{2j}) b_{2j-1}.$$

By definition of $[f]_\alpha$ and $[g]_\beta$,

$$|a_{2j-1} - a_{2j}| = \Big| f\big(2^{-n-1}(2j-1)\big) - f\big(2^{-n}j\big) \Big| \le [f]_\alpha \cdot \big(2^{-n-1}\big)^\alpha, \quad |b_{2j-1}| \le [g]_\beta \cdot \big(2^{-n-1}\big)^\beta.$$

Hence

$$|S_{n+1} - S_n| \le 2^n \cdot [f]_\alpha \cdot [g]_\beta \cdot \big(2^{-n-1}\big)^{\alpha+\beta} \le [f]_\alpha \cdot [g]_\beta \cdot 2^{-n(\alpha+\beta-1)}.$$

Since $\alpha + \beta > 1$, the series $\sum |S_{n+1} - S_n| < \infty$, and there exists

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left[S_1 + \sum_{j=1}^{n-1} (S_{j+1} - S_j) \right].$$

#4 (Jensen's Inequality, #42d, p.109). Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) = 1$, and let g be a function in $L^1(\mu)$. Show that for any convex function F on $\mathbb{R}^1$, we have

$$F\left(\int_X g \, d\mu \right) \le \int_X F(g) \, d\mu.$$

Hint. You can use without prove the fact that any convex function can be represented as an upper bound of linear functions:

$$F(u) = \sup_{\alpha \in A} (k_\alpha u + b_\alpha).$$

Proof. For each $\alpha \in A$, we have

$$k_\alpha \int_X g \, d\mu + b_\alpha = \int_X (k_\alpha u + b_\alpha) \, d\mu \le \int_X F(g) \, d\mu.$$

Using the above representation with $u := \int_X g \, d\mu$, we obtain the desired inequality.