Math 8602: REAL ANALYSIS. Spring 2016

Homework #1. Problems and Solutions.

#1. Show that the function

$$f(x)=\sum_{k=1}^{\infty}\frac{\sin(4^k x)}{2^k}$$

is continuous on $\mathbb{R}^1$, but its total variation $V[f;\,a,b]=\infty$ for any $a<b$.

Proof. The function f is continuous as the uniform limit of continuous functions:

$$f(x)=\lim_{n\to\infty}S_n(x),\quad \text{where}\quad S_n(x):=\sum_{k=1}^{n}2^{-k}\sin(4^k x).$$

Further, we fix $a<b$ and an interval $I\subset(a,b)$ of length $l=4^{-n}\cdot 2\pi$. Note that the function

$$f_n(x):=\sum_{k=n}^{\infty}2^{-k}\sin(4^k x)\quad \text{is}\quad l\text{-periodic},\quad \text{i.e.}\quad f_n(x+l)\equiv f_n(x).$$

Since $f=S_{n-1}+f_n$, where S_{n-1} is a smooth function, it suffices to show that

$$V:=V[f_n;\,I]:=\quad(\text{the total variation of } f_n \text{ on } I)\quad=\infty.$$

Suppose otherwise, i.e. $V<\infty$, and let μ_n be a signed measure on I which corresponds to f_n according to Theorem 3.29. Then for integers $m\ge n$,

$$V=|\mu_n|\,(I)\ge\int_I\big|\cos(4^m x)\big|\,d|\mu_n|\ge\int_I\cos(4^m x)\,d\mu_n=:\int_I\cos(4^m x)\,df_n.$$

Since both f_n and $\cos(4^m x)$ are l-periodic, using integration by parts (Theorem 3.36), we see that the boundary terms cancel, hence

$$V\ge-\int_I f_n\,d\cos(4^m x)=4^m\int_I f_n\,\sin(4^m x)\,dx.$$

Further, for $k,m\ge n$, we have

$$\sin(4^k x)\cdot\sin(4^m x)=\frac{1}{2}\left[\cos(4^k-4^m)x-\cos(4^k+4^m)x\right].$$

By periodicity, the integrals of these expressions over I are zeros for $k\ne m$. Therefore,

$$V\ge 4^m\sum_{k=n}^{\infty}2^{-k}\int_I\sin(4^k x)\cdot\sin(4^m x)\,dx=2^m\int_I\sin^2(4^m x)\,dx=2^{m-1}l=2^m\cdot\pi 4^{-n}.$$

For large m, we have a contradiction with the assumption $V<\infty$. Therefore, $V=\infty$.

#2. (Problem 36 on p. 127). Let X be the set of all real-valued Lebesgue measurable functions f on $[0,1]$ satisfying the inequality $|f|\le 1$. Show that there is NO topology $\mathcal{T}$ on X such that $f_n\to 0$ a.e. as $n\to\infty$ if and only if it converges with respect to $\mathcal{T}$.

Proof. Suppose there is such a topology $\mathcal{T}$. Let G be an open set ($G\in\mathcal{T}$) containing the function $f\equiv 0$. We claim that

$$\exists\varepsilon>0\quad\text{such that from}\quad g\in X\quad\text{and}\quad ||g||_1:=\int_0^1|g(x)|\,dx<\varepsilon\quad\text{it follows}\quad g\in G. \tag{1}$$

Indeed, otherwise we have: $\forall n \in \mathbb{N}, \quad \exists g_n \notin G$ with $||g_n||_1 < 1/n$. By Theorem 2.30, $\exists$ a subsequence $g_{n_j} \to 0$ a.e. By our assumption, $g_{n_j} \to 0$ with respect to $\mathcal{T}$, which implies that $g_{n_j} \in G$ for large enough j. However, $g_n \notin G$ for all n. This contradiction proves (1).

Now take a sequence $f_n \in X$ such that $||f_n||_1 \to 0$, but f_n does not converge to 0 a.e. For example, one can take f_n from iv on p. 61. From (1) it follows that for an arbitrary open set G containing $f \equiv 0$, there exists n_0 such that $f_n \in G$, $\forall n \ge n_0$. This means that $f_n \to 0$ with respect to $\mathcal{T}$. Since f_n does not converge to 0 a.e., these two kinds of convergence are not equivalent.

#3. Let f be a real valued continuous function on $\mathbb{R}^1$ such that $f(x) \equiv 0$ for $|x| \ge 2$.
Show that

$$f^{(\varepsilon)}(x) := \int\limits_{\mathbb{R}^1} f(x - \varepsilon y)\varphi(y)dy \ \to \ f(x) \quad \text{as} \quad \varepsilon \searrow 0$$

uniformly on $\mathbb{R}^1$, where

$$\varphi(y) := \frac{1}{\sqrt{\pi}} \cdot e^{-y^2}.$$

Proof. Since f is continuous on $[-2, 2]$, it is bounded: $|f| \le M = \text{const} < \infty$, and uniformly continuous:

$$\omega(\rho) := \sup_{|x-y| \le \delta} |f(x) - f(y)| \to 0 \quad \text{as} \quad \delta \searrow 0.$$

For an arbitrary constant $A > 0$, we can write

$$\big|f^{(\varepsilon)}(x) - f(x)\big| = \left| \int\limits_{\mathbb{R}^1} \big[f(x - \varepsilon y) - f(x)\big]\, \varphi(y)\, dy \right| \le \left(\int\limits_{|y| \le A} + \int\limits_{|y| > A} \right) \big|f(x - \varepsilon y) - f(x)\big|\, \varphi(y)\, dy$$

$$\le \omega(A\varepsilon) + 2M \cdot c_A, \quad \text{where} \quad c_A := \int\limits_{|y| > A} \varphi(y)\, dy;$$

$$\limsup_{\varepsilon \searrow 0} \sup_{\mathbb{R}^1} |f^{(\varepsilon)} - f| \le 2M \cdot c_A \to 0 \quad \text{as} \quad A \to \infty.$$

This implies the uniform convergence $f^{(\varepsilon)} \to f$ as $\varepsilon \searrow 0$ uniformly on $\mathbb{R}^1$.

#4. Use the previous problem for the proof of the Weierstrass theorem: every continuous function on $[-1, 1]$ can be uniformly approximated by polynomials.

Proof. Obviously, every function $f \in C([-1, 1])$ can be extended as a continuous function on $\mathbb{R}^1$ satisfying $f(x) \equiv 0$ for $|x| \ge 2$. By the previous problem, it suffices to show that the function $f^{(\varepsilon)}$ can be uniformly approximated by polynomials. Using substitution $z = x - \varepsilon y$, $y = \varepsilon^{-1}(x - z)$, we can rewrite the expression for $f^{(\varepsilon)}$ in the form

$$f^{(\varepsilon)}(x) = \varepsilon^{-1} \int\limits_{|z| \le 2} f(z)\, \varphi\big(\varepsilon^{-1}(x - z)\big)\, dz.$$

For $|x| \le 1$, $|z| \le 2$, we have $|y| \le 3/\varepsilon$. Fix an arbitrarily small $\delta > 0$. Note that the corresponding Taylor polynomials $\varphi_n(y) \to \varphi(y)$ as $n \to \infty$ uniformly on $|y| \le 3/\varepsilon$. Choose a large n such that

$$\sup_{|y| \le 3/\varepsilon} |\varphi_n - \varphi| \le \frac{\varepsilon\delta}{4M}, \quad \text{where} \quad M := \sup |f|.$$

Then

$$P_n(x) := \varepsilon^{-1} \int\limits_{|z| \le 2} f(z)\, \varphi_n\big(\varepsilon^{-1}(x - z)\big)\, dz.$$

is a polynomial of degree $\le 2n$ satisfying

$$\big|f^{(\varepsilon)}(x) - P_n(x)\big| \le \varepsilon^{-1} \int\limits_{|z| \le 2} |f(z)| \cdot \Big|(\varphi_n - \varphi)\big(\varepsilon^{-1}(x - z)\big)\Big|\, dz \le \varepsilon^{-1} \int\limits_{|z| \le 2} M \cdot \frac{\varepsilon\delta}{4M}\, dz = \delta.$$

for $|x| \le 1$. This proves the desired property.