Math 8602: REAL ANALYSIS. Spring 2016

Homework #5. Problems and Solutions.

#1. Let f be a function in $L^1(\mathbb{R}^1)$. Show that

$$\int\limits_{\mathbb{R}^1} f(x)\sin(\omega x)\,dx \ \to 0 \quad \text{as} \quad \omega \to \infty.$$

Proof. This problem is very similar to Problem 3 on Final Exam in the previous semester. By Theorem 2.26, every function $f \in L^1(\mathbb{R}^1)$ can be approximated in $L^1(\mathbb{R}^1)$ by functions $g \in C_0(\mathbb{R}^1)$ – continuous functions with compact support. In turn, by the Dominated Convergence Theorem, every function $g \in C_0(\mathbb{R}^1)$ can be approximated in $L^1(\mathbb{R}^1)$ by functions

$$g_h(x) := \frac{1}{h}\int\limits_x^{x+h} g(y)\,dy \ \in \ (C^1 \cap C_0)(\mathbb{R}^1),$$

i.e. the L^1-norms $||g_h - g||_1 \to 0$ as $h \searrow 0$. Therefore, $\forall f \in L^1(\mathbb{R}^1)$ and $\forall \varepsilon > 0$, $\exists g_h \in (C^1 \cap C_0)(\mathbb{R}^1)$ with $||g_h - f||_1 \le \varepsilon$. We can write

$$I(\omega) := \int\limits_{\mathbb{R}^1} f(x)\sin(\omega x)\,dx = I_1(\omega) + I_2(\omega),$$

where

$$\begin{aligned} I_1(\omega) \ &:= \int\limits_{\mathbb{R}^1} \big[f(x) - g_h(x)\big]\sin(\omega x)\,dx, \quad |I_1(\omega)| \le ||g_h - f||_1 \le \varepsilon, \\ I_2(\omega) \ &:= \int\limits_{\mathbb{R}^1} g_h(x)\sin(\omega x)\,dx, \quad |I_2(\omega)| \overset{\text{(by parts)}}{=} \frac{1}{\omega}\cdot\left|\int\limits_{\mathbb{R}^1} g_h'(x)\cos(\omega x)\,dx\right| \le \frac{1}{\omega}\cdot ||g_h'||_1 \to 0 \quad \text{as} \quad \omega \to \infty. \end{aligned}$$

Then $\limsup\limits_{\omega\to\infty} |I(\omega)| \le \varepsilon$, and since $\varepsilon > 0$ can be taken arbitrarily small, we get $I(\omega) \to 0$ as $\omega \to \infty$.

#2. Let $f(x) \in L^1_{loc}(\mathbb{R})$ and

$$f\Big(\frac{x+y}{2}\Big) \le \frac{f(x) + f(y)}{2} \qquad \text{for all} \quad x, y \in \mathbb{R}.$$

Show that f is convex on $\mathbb{R}$.

Proof. This statement is true under a more general assumption that $|f| < \infty$ a.e. The convexity of f means that a portion of the graph of $y = f(x)$ between two arbitrary point x_1 and x_2 in $\mathbb{R}$ lies below the segment connecting the point $(x_1, f(x_1))$ and $(x_2, f(x_2))$ in $\mathbb{R}^2$. By a linear transform, the proof of this fact is reduced to the case $x_1 = -1$, $x_2 = 1$, and $f(-1) = f(1) = 0$; in this case we must have $f(x) \le 0$ on $[-1, 1]$.

Suppose otherwise, i.e. $f(x_0) \ge a = \text{const} > 0$ for some $x_0 \in (-1, 1)$. Take a small $h_0 > 0$, such that $[x_0 - h_0, x_0 + h_0] \subseteq [-1, 1]$. By our assumptions,

$$0 < a \le f(x_0) \le \frac{f(x_0 + h) + f(x_0 - h)}{2}, \qquad \forall h \in [-h_0, h_0].$$

For such h, either $f(x_0 + h) \ge a$ or $f(x_0 - h) \ge a$. In other words,

$$[-h_0, h_0] = A \cup (-A), \quad \text{where} \quad A := \{h \in [-h_0, h_0] : \quad f(x_0 + h) \ge a\}.$$

Then the set $E(a) := [-1, 1] \cap \{f \ge a > 0\}$ contains $x_0 + A$, and its Lebesgue measure

$$m\big(E(a)\big) \ge m(A) = \frac{1}{2}\cdot\big(m(A) + m(-A)\big) \ge \frac{1}{2}\cdot m\big(A \cup (-A)\big) = \frac{1}{2}\cdot m\big([-h_0, h_0]\big) = h_0 > 0.$$

On the other hand,

$$E(a) = E^-(a) \cup E^+(a), \quad \text{where} \quad E^-(a) := E(a) \cap [-1,0], \quad E^+(a) := E(a) \cap [0,1],$$

so that $m\big(E^-(a)\big) + m\big(E^+(a)\big) = m\big(E(a)\big) \ge h_0 > 0$. We can assume that $m\big(E^-(a)\big) \ge h_0/2$ (replacing $f(x)$ by $f(-x)$ if necessary). By our condition, we always have

$$2f(x) \le f(-1) + f(1+2x) = f(1+2x).$$

Introducing a linear map $T(x) := 1 + 2x$, we see that

$$T\big(E^-(a)\big) \subseteq E(2a), \quad \text{and} \quad m\big(E(2a)\big) \ge m\Big(T\big(E^-(a)\big)\Big) = 2 \cdot m\big(E^-(a)\big) \ge h_0 > 0.$$

Here the key observation is that from $m\big(E(a)\big) \ge h_0 > 0$ it follows $m\big(E(2a)\big) \ge h_0 > 0$. By iteration,

$$m\big(E(2^k a)\big) := m\big([-1,1] \cap \{f \ge 2^k a\}\big) \ge h_0 > 0, \qquad \forall k = 1, 2, \ldots.$$

Since $2^k a \nearrow +\infty$ as $k \to \infty$, and $|f| < \infty$ a.e., we get a desired contradiction.

#3. Show that

$$H_n(x) := (-1)^n e^{x^2} \left(e^{-x^2}\right)^{(n)}$$

are polynomials of degree n (the *Hermite* polynomials) satisfying

$$\int_{-\infty}^{\infty} e^{-x^2} H_k H_n \, dx = 0 \quad \text{for} \quad k \ne n.$$

Derive the equality

$$F(t,x) := \sum_{n=0}^{\infty} \frac{t^n}{n!} \cdot H_n(x) = e^{2tx - t^2}.$$

Proof. It is easy to see that $H_n' = 2xH_n - H_{n+1}$, and by induction, H_n is a polynomial of degree n for every n. Since $H_k^{(n)} = 0$ for $n > k$, integrating by parts implies

$$\int_{-\infty}^{\infty} e^{-x^2} H_k H_n \, dx = \int_{-\infty}^{\infty} H_k \cdot (-1)^n \left(e^{-x^2}\right)^{(n)} dx = \int_{-\infty}^{\infty} H_k^{(n)} e^{-x^2} dx = 0.$$

By symmetry, this equality also holds true for $n < k$. Finally, using the Taylor expansion

$$f(x+h) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x)}{n!} \cdot h^n \quad \text{with} \quad f(x) := e^{-x^2}, \quad h := -t,$$

we get

$$F(t,x) = e^{x^2} e^{-(x-t)^2} = e^{2tx - t^2}.$$

#4. Let $\{x_n\}$ be a sequence in a Hilbert space $\mathcal{H}$ such that $||x_n|| \le 1$ for all n, and for each $y \in \mathcal{H}$, we have $(x_n, y) \to 0$ as $n \to \infty$. Show that there is a subsequence $\{x_{n_j}\}$ such that

$$\frac{1}{k} \cdot \left(x_{n_1} + \cdots + x_{n_k}\right) \to 0 \quad \text{as} \quad k \to \infty.$$

Proof. Take $n_1 = 1$, and then for $j = 2, 3, \ldots$, choose n_j such that

$$|(x_{n_i}, x_{n_j})| \le \frac{1}{j^2} \quad \text{for all} \quad i < j.$$

Then $y_k := \frac{1}{k} \cdot \left(x_{n_1} + \cdots + x_{n_k}\right)$ satisfy

$$||y_k||^2 = (y_k, y_k) = \frac{1}{k^2} \sum_{i=1}^{k} ||x_{n_j}||^2 + \frac{2}{k^2} \sum_{1 \le i < j \le k} (x_{n_i}, x_{n_j}) \le \frac{1}{k} + \frac{2}{k^2} \sum_{j=1}^{k} \frac{1}{j} \le \frac{3}{k} \to 0 \quad \text{as} \quad k \to \infty.$$