# On the boundary value problems for fully nonlinear elliptic equations of second order

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**Abstract**

Fully nonlinear second-order, elliptic equations $F(x, u, Du, D^2u) = 0$ are considered in a bounded domain $\Omega \subset R^n, n \geq 2$. The class of equations includes the Bellman equations $\sup_m (L^m u + f^m) = 0$, where the functions $f^m$ and the coefficients of the linear operators $L^m$ are bounded in the Hölder space $C^\alpha(\overline{\Omega}), 0 < \alpha < 1$. We prove the interior $C^{2,\alpha}$-smoothness of solutions in $\Omega$ with some small $\alpha > 0$. Under the Dirichlet boundary condition $u = \varphi$ on $\partial\Omega$ with $\varphi \in C^{2,\alpha}(\overline{\Omega})$ and $\partial\Omega \in C^{2,\alpha}$, the solutions $u \in C^{2,\alpha}(\overline{\Omega})$. Under the oblique derivative condition $b_0 u + b \cdot Du = \varphi$ on $\partial\Omega$, where $b = (b_1, \cdots, b_n)$ is not tangent to $\partial\Omega$, the solutions $u \in C^{2,\alpha}(\overline{\Omega})$ if $b_i, \varphi \in C^{1,\alpha}(\overline{\Omega})$, and also $\partial\Omega \in C^{1,\alpha}$.

## Contents

## 1. Introduction

In this paper we consider general nonlinear elliptic equations including the *Bellman equations*

$$\sup_m (L^m u + f^m) = \sup_m (a_{ij}^m D_{ij} u + b_i^m D_i u + c^m u + f^m) = 0 \tag{1.1}$$

(the summation over repeated indices is everywhere understood). Such equations are also important from the viewpoint of the applications to the theory of controlled diffusion processes (see [10]).

We investigate the Dirichlet and the oblique derivative problems in a bounded domain $\Omega \subset R^n, n \geq 2$, for nonlinear elliptic equations in the Hölder space $C^{2,\alpha}(\overline{\Omega}), 0 < \alpha < 1$. Leaving aside the simpler one- and two- dimensional cases, we note that the interior $C^{2,\alpha}$-smoothness of solutions to the equation (1.1) was first proved in 1977 by Brézis and Evans [2] in the case when $m$ assumes only two values. In 1981, Krylov [11], [12] has established the $C^{2,\alpha}$-smoothness of the solutions of the Bellman elliptic and parabolic equations in the higher-dimensional case, both in the interior of the domain and near its boundary, under appropriate smoothness of the boundary and the boundary values of the solutions. At about the same time, Evans [7] (see also [9]) independently proved the $C^{2,\alpha}$-smoothness of solutions of elliptic equations (1.1) in the interior of the domain. Under the oblique derivative condition, the $C^{2,\alpha}$-smoothness of solutions near the boundary was proved in [16], [17]. In all those papers, and also in [4], [5], [9], [13], as they apply to (1.1), it is assumed that the functions $a_{ij}^m, b_i^m, c^m, f^m$ are uniformly bounded, together with all their first and second derivatives.

The $C^{2,\alpha}$-estimates of solutions fo the Bellman equation (1.1) with coefficients $a_{ij}^m, b_i^m, c^m, f^m$ in $C^\alpha$ for some small $\alpha \in (0,1)$, were first obtained in [24], including the estimates near the boundary in the Dirichlet case. For the oblique derivative problem, the corresponding result was proved in [1]. Some other extensions of the results in [24], both for the Dirichlet and for the oblique derivative problem, were derived by Trudinger [28]. There are also some close results in the papers of Caffarelli [3] and Wang [29], where they treat both the $C^{2,\alpha}$-estimates and the estimates in the Sobolev spaces $W^{2,p}$ when $f^m \in \mathcal{L}^p, 1 < p < \infty$.

Here we give an enlarged exposition of results in [24], [26], [1]. We introduce a class of nonlinear equations including the Bellman equaitons (1.1) with coefficients in $C^\alpha$ for some small $\alpha \in (0,1)$, and we show that the Dirichlet problem and the oblique derivative problem are solvable in $C^{2,\alpha}$. Under minimal assumptions on the boundary and the boundary data, we receive also the $C^{2,\alpha}$-estimates near the boundary for solutions of these problems. These results are formulated in Section 3 (Theorems 3.1–3.3). Notice that in the case of linear elliptic equations, they turn into the classical Schauder-type results (see [9], Ch.6; [19], Ch.3), even with an improvement: Theorem 3.3 states that the solution of the oblique derivative problem in $\Omega$ still belongs to $C^{2,\alpha}(\overline{\Omega})$ if $\partial\Omega \in C^{1,\alpha}$. For linear elliptic equations such an improvement was proved by Lieberman [15].

The most essential part in the proof of Theorems 3.1–3.3 consists in the appropriate a priori $C^{2,\alpha}$-estimates of the solutions. The basic idea of deriving such estimates is the "local" decomposition of the solution into "smooth" and "small" terms. The "smooth" term is the solution of an auxiliary problem for the simplest nonlinear equation corresponding to the case when in (1.1) we have $a_{ij}^m, f^m = const$, and $b_i^m = c^m = 0$. For the technical realization of this idea, it is convenient to use some equivalent seminorms in $C^{2,\alpha}$ introduced by Campanato [6]. In Section 2, we expose a simple approach to the Campanato type of seminorms which are equivalent to the usual seminorms in "weighted" Hölder spaces (Theorem 2.1).

For the completeness of the presentation, in Section 4 we prove the interior $C^{2,\alpha}$-estimates of Krylov [11] and Evans [7], in a particular case of the simplest nonlinear equaitons. Our approach is new in some details, while it relies, as well as [11], [7], on the results of [14], [22]. In Section 5, we extend the interior $C^{2,\alpha}$-estimates to the solutions of the general nonlinear equaitons. The next Section 6 is devoted to the boundary behaviour of solutions to the linear elliptic equations with measurable coefficients. These auxiliary results help us to get the $C^{2,\alpha}$-estimates near the boundary for soluitons of nonlinear equations; the Dirichlet and the oblique derivative conditions are treated in Sections 7 and 8 correspondingly.

BASIC NOTATIONS. $R^n$ is Euclidean space of dimension $n$, with standard basis $\{e_1, \cdots, e_n\}$, and points $x = (x_1, \cdots, x_n)$ written in coordinates relative to this basis; $(x, y) = x_i y_i$ is the inner product of $x, y \in R^n$; $|x| = (x, x)^{1/2} = (\sum x_i^2)^{1/2}$; $R_+^n = \{x \in R^n : x_n > 0\}$, $R_0^n = \{x \in R^n : x_n = 0\}$, $S^n$ denotes the $n(n+1)/2$- dimensional space of all real symmetric $n \times n$ matrices. We will identify $R_0^n$ and $R^{n-1}$.

$\partial\Omega$ is the boundary of the set $\Omega \subset R^n$, $\overline{\Omega} = \Omega \cup \partial\Omega$; $B_\rho(x) = \{y \in R^n : |y - x| < \rho\}$ is the ball of radius $\rho > 0$ centered at $x \in R^n$,

$$B_\rho^+(x) = R_+^n \cap B_\rho(x), \quad B_\rho^0(x) = R_0^n \cap B_\rho(x), \quad \Omega_\rho(x) = \Omega \cap B_\rho(x).$$

$l = (l_1, \cdots, l_n)$ is a multi-index, i.e. $l_i = integer \geq 0$, with $|l| = \sum l_i$. We define $x^l = x_1^{l_1} x_2^{l_2} \cdots x_n^{l_n}$, $l! = l_1! \, l_2! \cdots l_n!$. For functions $u = u(x)$, we set $D_i u = \partial u/\partial x_i$, $D_{ij} u = \partial^2 u/\partial x_i \partial x_j$; $Du = (D_1 u, \cdots, D_n u)$ is the gradient of $u$, $D^2 u = [D_{ij} u]$ is the Hessian matrix. Moreover, we define the first and the second derivatives of $u$ in the direction $\lambda \in R^n$ as follows: $D_\lambda u = \partial u/\partial \lambda = \lambda_i D_i u$, $D_{\lambda\lambda} u = \partial^2 u/\partial \lambda^2 = \lambda_i \lambda_j D_{ij} u$. We will also use the multi-index notation $D^l u = \partial^{|l|} u/\partial x_1^{l_1} \cdots \partial x_n^{l_n}$, with the understanding $D^0 u = u$. For integer $k \geq 0$, $\mathcal{P}_k$ denotes the collection of all polynomials of degree at most $k$. In particular, the *Taylor polynomial* of degree $k$ for the function $u$ at the point $y \in R^n$ is

$$T_{y,k}\, u(x) = \sum_{|l| \leq k} D^l u(y) \cdot (x - y)^l / l! \in \mathcal{P}_k. \tag{1.2}$$

Throughout this paper, $N$ will denote various positive constants. In the intermediate calculations, we will usually omit the dependence of $N$ on the original quantities.

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## 2. The Hölder spaces

Let $\Omega$ be a domain in $R^n$, $n \geq 1$. For $k = 0, 1, 2, \cdots$, we denote $C^k(\Omega)$ the set of functions $u = u(x)$ whose derivatives $D^l u$ for $|l| \leq k$ are continuous in $\Omega$. We set

$$(2.1) \qquad |u|_0 = |u|_{0;\Omega} = \sup_{\Omega} |u|, \quad [u]_{k,0} = [u]_{k,0;\Omega} = \max_{|l|=k} |D^l u|_{0;\Omega}.$$

**Definition 2.1.** $C^{k,0}(\Omega)$ is the Banach space of functions $u \in C^k(\Omega)$ with the finite norm

$$(2.2) \qquad |u|_k = |u|_{k,0} = |u|_{k,0;\Omega} = \sum_{j=0}^{k} [u]_{j,0;\Omega}, \quad k = 0, 1, 2, \cdots .$$

Further, we call $u$ *Hölder continuous with exponent* $\alpha$ *in* $\Omega$, if the quantity

$$(2.3) \qquad [u]_\alpha = [u]_{\alpha;\Omega} = \sup_{x,y\in\Omega} |u(x) - u(y)|/|x - y|^\alpha, \quad 0 < \alpha \leq 1$$

is finite. We set

$$(2.4) \qquad [u]_{k,\alpha} = [u]_{k,\alpha;\Omega} = \max_{|l|=k} [D^l u]_{\alpha;\Omega}.$$

**Definition 2.2.** The *Hölder space* $C^{k,\alpha}(\Omega)$ is the Banach space of functions $u \in C^k(\Omega)$ with the finite norm

$$(2.5) \qquad |u|_{k,\alpha} = |u|_{k,\alpha;\Omega} = |u|_{k,0;\Omega} + [u]_{k,\alpha;\Omega}, \quad k = 0, 1, 2, \cdots , \quad 0 < \alpha \leq 1.$$

We will also use the similar notations for closed domains $\overline{\Omega}$ and more generally, for $\Omega \cup \Gamma$, where $\Gamma \subset \partial\Omega$. Obviously, for bounded domain $\Omega$ we have $C^{k,0}(\overline{\Omega}) = C^k(\overline{\Omega})$. For simplicity we will write $C^{0,\alpha} = C^\alpha$, if $0 < \alpha < 1$. From the elementary inequality

$$|u(x)v(x) - u(y)v(y)| \leq |u(x)| \cdot |v(x) - v(y)| + |v(y)| \cdot |u(x) - u(y)|$$

and (2.1), (2.3), it follows

$$(2.6) \qquad [uv]_\alpha \leq |u|_0 \cdot [v]_\alpha + |v|_0 \cdot [u]_\alpha \ \text{ for } u, v \in C^\alpha(\Omega), \quad 0 < \alpha \leq 1.$$

The following lemma contains the well known *interpolation inequalities* (see [9], Sec. 6.8).

**Lemma 2.1.** *Suppose $j+\beta < k+\alpha$, where $j,k = 0,1,2,\cdots$, and $0 \le \alpha,\beta \le 1$. Let $u \in C^{k,\alpha}(B_r)$, where $B_r = B_r(x_0), r > 0$. Then for any $\varepsilon > 0$ we have*

$$r^{j+\beta}[u]_{j,\beta;B_r} \le \varepsilon\, r^{k+\alpha}[u]_{k,\alpha;B_r} + N(\varepsilon)|u|_{0;B_r}, \tag{2.7}$$

*with a constant $N(\varepsilon) = N(\varepsilon,n,k,\alpha,\beta)$. The similar inequalities are also true for $B_r^+ = B_r^+(x_0), x_0 \in \overline{R^n_+} = \{x \in R^n : x_n \ge 0\}$.*

Further, let a subset $\Gamma \subset \partial\Omega$ be given, $\Gamma \ne \partial\Omega$ (the case $\Gamma = \emptyset$ is not excluded). For $k = 0,1,2,\cdots$, $0 \le \alpha \le 1$, $\gamma \in R^1$, and $u \in C^k(\Omega\cup\Gamma)$, we set

$$[u]^{(\gamma)}_{k,\alpha} = [u]^{(\gamma)}_{k,\alpha;\Omega\cup\Gamma} = \sup_{x\in\Omega\cup\Gamma} d^{k+\alpha+\gamma}(x)\cdot[u]_{k,\alpha;\Omega(x)}, \tag{2.8}$$

where

$$d(x) = \frac{1}{2}\mathrm{dist}(x,\partial\Omega\setminus\Gamma), \quad \Omega(x) = \Omega_{d(x)}(x) = \Omega\cap B_{d(x)}(x). \tag{2.9}$$

**Definition 2.3.** For $\Gamma \subset \partial\Omega$, $k = 0,1,2,\cdots$, and $\gamma \in R^1$, $C^{k;\gamma}(\Omega\cup\Gamma) = C^{k,0;\gamma}(\Omega\cup\Gamma)$ is the Banach space of functions $u \in C^k(\Omega\cup\Gamma)$ with the finite norm

$$\|u\|^{(\gamma)}_{k,0} = \|u\|^{(\gamma)}_{k,0;\Omega\cup\Gamma} = \sum_{j=0}^{k}[u]^{(\gamma)}_{j,0;\Omega\cup\Gamma}. \tag{2.10}$$

**Definition 2.4.** For $\Gamma \subset \partial\Omega$, $k = 0,1,2,\cdots$, $0 < \alpha \le 1$, and $\gamma \in R^1$, the *weighted Hölder space* $C^{k,\alpha;\gamma}(\Omega\cup\Gamma)$ is the Banach space of functions $u \in C^k(\Omega\cup\Gamma)$ with the finite norm

$$\|u\|^{(\gamma)}_{k,\alpha} = \|u\|^{(\gamma)}_{k,\alpha;\Omega\cup\Gamma} = \|u\|^{(\gamma)}_{k,0;\Omega\cup\Gamma} + [u]^{(\gamma)}_{k,\alpha;\Omega\cup\Gamma}. \tag{2.11}$$

We will consider only very special cases of $\Gamma$: either $\Gamma = \emptyset$ and $\Omega$ is an arbitrary domain in $R^n$, or $\Gamma \subset \partial\Omega\cap R^n_0$ and $\Omega \subset R^n_+$. Therefore, in (2.9) we have either $\Omega(x) = B_{d(x)}(x)$ or $\Omega(x) = B^+_{d(x)}(x), x \in \overline{R^n_+}$. So we can rewrite (2.7) in the form

$$d^{j+\beta}(x)[u]_{j,\beta;\Omega(x)} \le \varepsilon\, d^{k+\alpha}(x)[u]_{k,\alpha;\Omega(x)} + N(\varepsilon)|u|_{0;\Omega(x)}.$$

Multiplying both sides of this onequality by $d^\gamma(x)$, and then taking the *sup* over $x \in \Omega\cup\Gamma$, we arrive at the following interpolation inequalities for weighted Hölder spaces.

**Corollary 2.1.** *Suppose $j+\beta < k+\alpha$, and let $u \in C^{k,\alpha;\gamma}(\Omega\cup\Gamma)$, $\gamma \in R^1$. Then for and $\varepsilon > 0$ we have*

$$[u]^{(\gamma)}_{j,\beta;\Omega\cup\Gamma} \le \varepsilon\,[u]^{(\gamma)}_{k,\alpha;\Omega\cup\Gamma} + N(\varepsilon,n,k,\alpha,\beta)\cdot|u|^{(\gamma)}_{0,0;\Omega\cup\Gamma}. \tag{2.12}$$

The following lemma is related to the approximation of a function $u$ by means of its Taylor polynomial $T_{y,k}u$ defined in (1.2).

**Lemma 2.2.** *Let $u \in C^{k,\alpha}(\overline{\Omega})$, $0 < \alpha \le 1$. Then for any $x, y \in \overline{\Omega}$ such that the segment $[x, y] \subset \overline{\Omega}$, we have*

$$|u(x) - T_{y,k}\, u(x)| \le N(n)[u]_{k,\alpha} \cdot |x - y|^{k+\alpha}. \tag{2.13}$$

*Proof:* By Taylor's formula,

$$u(x) = T_{y,k-1}\, u(x) + \sum_{|l|=k} D^l u(\xi) \cdot (x - y)^l / l!,$$

where $\xi \in [x, y]$. Further, from (2.4) it follows

$$\max_{|l|=k} |D^l u(\xi) - D^l u(y)| \le [u]_{k,\alpha} \cdot |\xi - y|^\alpha \le [u]_{k,\alpha} \cdot |x - y|^\alpha.$$

Therefore,

$$|u(x) - T_{y,k}\, u(x)| = \left| \sum_{|l|=k} (D^l u(\xi) - D^l u(y)) \cdot (x - y)^l / l! \right| \le N[u]_{k,\alpha} \cdot |x - y|^{k+\alpha},$$

that completes the proof. □

**Corollary 2.2.** *Let $u \in C^{k,\alpha}(\Omega_\rho)$, where $\Omega_\rho = B_\rho(x_0), x_0 \in R^n$, or $\Omega_\rho = B^+_\rho(x_0), x_0 \in \overline{R^n_+}$. Then*

$$E_k[u; \Omega_\rho] = \inf_{p \in \mathcal{P}_k} \sup_{\Omega_\rho} |u - p| \le N(n)[u]_{k,\alpha}\, \rho^{k+\alpha}. \tag{2.14}$$

**Lemma 2.3.** *Let $k = 0, 1, 2, \cdots$, $0 < \alpha \le 1$, and $u \in C^{k,\alpha}(B_\rho)$, $B_\rho = B_\rho(x_0)$. Then for any $\varepsilon > 0$ we have*

$$\rho^{-\alpha} \max_{|l|=k} \operatorname*{osc}_{B_\rho} D^l u \;\le\; \varepsilon[u]_{k,\alpha;B_\rho} + N(\varepsilon, n, k, \alpha) \cdot \rho^{-k-\alpha} E_k[u; B_\rho], \tag{2.15}$$

*where* $\operatorname{osc} f = \sup f - \inf f$. *The similar inequalities are also true for $B^+_\rho = B^+_\rho(x_0), x_0 \in \overline{R^n_+}$.*

*Proof:* Using the elementary inequality $\operatorname{osc} f \le 2 \sup |f|$ and (2.7) with $r = \rho$, $j = k$, $\beta = 0$, we have

$$\frac{1}{2}\rho^{-\alpha} \max_{|l|=k} \operatorname*{osc}_{B_\rho} D^l u \;\le\; \rho^{-\alpha}[u]_{k,0;B_\rho} \;\le\; \varepsilon[u]_{k,\alpha;B_\rho} + N(\varepsilon)\rho^{-k-\alpha} \sup_{B_\rho} |u|.$$

For arbitrary $p \in \mathcal{P}_k$, the left-hand side of this inequality and $[u]_{k,\alpha}$ remain the same if we replace $u$ by $u - p$. After the replacement, we take the infimum of the right-hand side over $p \in \mathcal{P}_k$. On redefining $\varepsilon$, this will give us the desired estimate. □

The next theorem is similar to Theorem 2.1 in [26] (see also [6]).

**Theorem 2.1.** *Let $k = 0, 1, 2, \cdots$, $0 < \alpha \le 1$, $\gamma \in R^1$, and $u \in C^k(\Omega \cup \Gamma)$ has a finite seminorm $[u]^{(\gamma)}_{k,\alpha}$ in (2.8). Set*

$$\omega_k(x, \rho) = \max_{|l|=k} \operatorname*{osc}_{\Omega_\rho(x)} D^l u, \quad \Omega_\rho(x) = \Omega \cap B_\rho(x), \tag{2.16}$$

$$\hat{M}^{(\gamma)}_{k,\alpha} = \hat{M}^{(\gamma)}_{k,\alpha}[u; \Omega \cup \Gamma] = \sup_{x \in \Omega \cup \Gamma} d^{k+\alpha+\gamma}(x) \sup_{\rho \in (0, d(x)]} \rho^{-\alpha} \omega_k(x, \rho), \tag{2.17}$$

$$M^{(\gamma)}_{k,\alpha} = M^{(\gamma)}_{k,\alpha}[u; \Omega \cup \Gamma] = \sup_{x \in \Omega \cup \Gamma} d^{k+\alpha+\gamma}(x) \sup_{\rho \in (0, d(x)]} \rho^{-k-\alpha} E_k[u; \Omega_\rho(x)], \tag{2.18}$$

*where $d(x) = \frac{1}{2}\operatorname{dist}(x, \partial\Omega \setminus \Gamma)$, $E_k$ is defined in (2.14). Then all the seminorms $[u]^{(\gamma)}_{k,\alpha}$, $\hat{M}^{(\gamma)}_{k,\alpha}$, and $M^{(\gamma)}_{k,\alpha}$ are equivalent :*

$$N_1^{-1}[u]^{(\gamma)}_{k,\alpha} \le \hat{M}^{(\gamma)}_{k,\alpha} \le N_2 [u]^{(\gamma)}_{k,\alpha}, \tag{2.19}$$

$$N_3^{-1}[u]^{(\gamma)}_{k,\alpha} \le M^{(\gamma)}_{k,\alpha} \le N_4 [u]^{(\gamma)}_{k,\alpha}, \tag{2.20}$$

*where the constants $N$ depend only on $n, k, \alpha, \gamma$.*

*Proof:* To prove the first inequality in (2.19), we fix $x_0 \in \Omega \cup \Gamma$, $d = d(x_0) = \frac{1}{2}\operatorname{dist}(x_0, \partial\Omega \setminus \Gamma)$, $|l| = k$, and $x, y \in \Omega_d(x_0)$, such that

$$[u]^{(\gamma)}_{k,\alpha} \le 2 d^{k+\alpha+\gamma} |D^l u(x) - D^l u(y)| / |x - y|^\alpha. \tag{2.21}$$

We consider separately the cases (a) $\rho = |x - y| < d/2$ and (b) $\rho \ge d/2$. In the case (a), we have $x, y \in \Omega_{d/2}(y) \subset \Omega_{3d/2}(x_0)$. Since $d/2 \le d(y) \le 3d/2$, from (2.21) it follows

$$[u]^{(\gamma)}_{k,\alpha} \le N\, d^{k+\alpha+\gamma}(z) \cdot \rho^{-\alpha} \operatorname*{osc}_{\Omega_\rho(z)} D^l u \tag{2.22}$$

with $N = N(k, \gamma)$, $z = y$. Obviously, this inequality is also true in the case (b) for $z = x_0, \rho = d$, and $N = 2^{1+\alpha}$. In any case, we have (2.22), where $0 < \rho \le d(z)$, that yields the first inequality in (2.19) with $N_1 = N_1(k, \gamma)$. The second inquality is trivial with $N_2 = 2^\alpha$.

Further, from Lemma 2.3 and the definition of $[u]^{(\gamma)}_{k,\alpha}$, $\hat{M}^{(\gamma)}_{k,\alpha}$, and $M^{(\gamma)}_{k,\alpha}$, we get the inequailty

$$\hat{M}^{(\gamma)}_{k,\alpha} \le \varepsilon [u]^{(\gamma)}_{k,\alpha} + N(\varepsilon, n, k, \alpha) \cdot M^{(\gamma)}_{k,\alpha}.$$

Setting $\varepsilon = (2N_1)^{-1}$ and using (2.19), we obtain the first inequality in (2.20). Finally, the last inequality follows immediately from Corollary 2.2 with $N_4 = N_4(n)$. □

We will investigate the boundary value problems in a bounded domain $\Omega$ under some natural restrictions on the boundary $\partial\Omega$. The following assumptions are usually called the *Lipschitz condition* on $\partial\Omega$.

**Assumptions 2.1.** There exist positive constants $r_0$ and $K_0$, such that for each $x_0 \in \partial\Omega$ we have

(2.23)
$$\Omega_{r_0}(x_0) = \Omega \cap B_{r_0}(x_0) = \{x = (x', x_n) \in R^n : x_n > \psi_0(x')\} \cap B_{r_0}(x_0)$$

in an orthonormal system centered at $x_0$, where $\psi_0$ is defined on the projection $B^0_{r_0}$ of $B_{r_0}(x_0)$ onto $R^n_0$, and

$$|\psi_0(x') - \psi_0(y')| \leq K_0 \cdot |x' - y'| \text{ for all } x', y' \in B^0_{r_0}. \tag{2.24}$$

**Lemma 2.4.** *Let $\Omega$ be a bounded domain in $R^n$, satisfying Assumptions 2.1, and let $0 < \alpha < 1$. Then for any function $f(x)$ on $\Omega$ with a finite seminorm $[f]^{(-\alpha)}_{1,0;\Omega}$, we have*

$$[f]_{\alpha;\Omega} \leq N(n, r_0, K_0, diam\,\Omega, \alpha) \cdot [f]^{(-\alpha)}_{1,0;\Omega}. \tag{2.25}$$

*Proof.* Let us fix $x^1, x^2 \in \Omega$, and set $r = |x^1 - x^2|$. From the geometrical properties of $\Omega$ it follows that we can choose $x_0 \in \Omega$ such that

$$B_{r/N}(x_0) \subset \Omega, \quad |x^k - x_0| \leq Nr, \quad k = 1, 2.$$

Further, we can connect $x_0$ with $x^k$ by means of a smooth path in $\Omega$,

$$\{x = h_k(s) : 0 \leq s \leq s_k\}, \quad h_k(0) = x_0, \quad h_k(s_k) = x^k,$$

parametrised by the arc length $s$ in such a manner that

(2.26)
$$0 \leq s_k \leq Nr, \quad d(h_k(s)) = \frac{1}{2}\text{dist}(h_k(s), \partial\Omega) \geq (s_k - s)/N, \quad 0 \leq s \leq s_k.$$

Since $\sup_\Omega d^{1-\alpha}(x) \max_i |D_i f(x)| \leq [f]^{(-\alpha)}_{1,0}$, we get

(2.27)
$$|f(x^k) - f(x_0)| \leq \int_0^{s_k} |Df(h_k(s))|\, ds \leq n[f]^{(-\alpha)}_{1,0} \cdot \int_0^{s_k} d^{\alpha-1}(h_k(s))\, ds.$$

By virtue of (2.26), the last integral does not exceed

$$N \int_0^{s_k} (s_k - s)^{\alpha-1}\, ds \leq N s_k^\alpha \leq N r^\alpha = N|x^1 - x^2|^\alpha,$$

so we obtain:

$$|f(x^1) - f(x^2)| \leq |f(x^1) - f(x_0)| + |f(x^2) - f(x_0)| \leq N[f]^{(-\alpha)}_{1,0} \cdot |x^1 - x^2|^\alpha.$$

This estimate with arbitrary $x^1, x^2 \in \Omega$ implies (2.25). □

**Remark 2.1.** In the standard approach to the Schauder interior estimates (see [9], Ch.6) the notation $[u]^{(\gamma)}_{k,\alpha}$ is used for

$$A = \max_{|l|=k} \sup_{x,y\in\Omega} d^{k+\alpha+\gamma}_{x,y} \frac{|D^l u(x) - D^l u(y)|}{|x - y|^\alpha} = \sup_{\delta>0} \delta^{k+\alpha+\gamma} [u]_{k,\alpha;\Omega_\delta}, \tag{2.28}$$

where $0 < \alpha \leq 1$, $k + \alpha + \gamma \geq 0$, $d_{x,y} = \text{dist}(\{x, y\}, \partial\Omega)$, and

$$\Omega_\delta = \{x \in \Omega : \text{dist}(x, \partial\Omega) > \delta\}. \tag{2.29}$$

In particular, in the case $\gamma = -k - \alpha$ we have $A = [u]_{k,\alpha;\Omega}$. One can show that for Lipschitz domains $\Omega$ seminorms $[u]_{k,\alpha}^{(\gamma)}$ in (2.8) and $A$ in (2.28) are equivalent, if $k + \alpha + \gamma \geq 0$.

However, in the case $k + \alpha + \gamma < 0$ we have $A < \infty$ only for polynomials of degree at most $k$ (and then $A = 0$), while $[u]_{k,\alpha}^{(\gamma)} < \infty$ for more general class of functions. For example, if $k + \alpha + \gamma < 0 \leq k + 1 + \gamma$ and $u \in C^{k+1}(\overline{B_1})$, then by the mean value theorem we obtain

$$[u]_{k,\alpha}^{(\gamma)} \leq N_1 [u]_{k+1,0}^{(\gamma)} \leq N_2 [u]_{k+1} < \infty.$$

## 3. Formulation of main existence results

Let $\Omega$ be a bounded domain in $R^n$ and constants $K, K_1 \geq 0, \nu \in (0, 1]$, $\alpha \in (0, 1)$ be fixed. We will consider nonlinear elliptic equations

$$F[u] = F(x, u, Du, D^2u) = 0 \text{ in } \Omega, \tag{3.1}$$

where $F(x, u, u_i, u_{ij})$ is defined on $\overline{\Omega} \times R^1 \times R^n \times S^n$ and satisfies the following conditions:

**Assumptions 3.1.** (F0) The function $F(x, u, u_i, u_{ij})$ is lower convex with respect to $[u_{ij}] \in S^n$; (F1) (the *ellipticity condition*)

$$\nu|\xi|^2 \leq F(x, u, u_i, u_{ij} + \xi_i\xi_j) - F(x, u, u_i, u_{ij}) \leq \nu^{-1}|\xi|^2 \text{ for all } \xi \in R^n;$$

(F2) $F(x, u, u_i, u_{ij})$ is nonincreasing with respect to $u$, and

$$|F(x, u, u_i, u_{ij}) - F(x, \overline{u}, \overline{u}_i, u_{ij})| \leq K \cdot (|u - \overline{u}| + \sum_i |u_i - \overline{u}_i|)$$

for all $x, u, \overline{u}, u_i, \overline{u}_i, u_{ij}$; (F3) $|F(x, 0, 0, 0)| \leq K_1$ for all $x \in \Omega$;
(F4) for any fixed $(u, u_i, u_{ij}) \in R^1 \times R^n \times S^n$, the seminorm in $C^\alpha$,

$$[F(\cdot, u, u_i, u_{ij})]_{\alpha;\Omega} \leq K \cdot (|u| + \sum_i |u_i| + \sum_{ij} |u_{ij}|) + K_1.$$

**Remark 3.1.** It is easy to see that if $F$ satisfies the additional condition
(F$^*$) $F(x, u, u_i, u_{ij})$ is infinitely differentiable with respect to $(u, u_i, u_{ij}) \in R^1 \times R^n \times S^n$, then conditions (F0)–(F2) can be rewritten as follows:
(F0$^*$) $\partial^2 F/\partial u_{ij}\partial u_{pq} \cdot \eta_{ij}\eta_{pq} \geq 0$ for all $[\eta_{ij}] \in S^n$; (F1$^*$) the functions $a_{ij} = \partial F/\partial u_{ij}$ satisfy

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \text{ for all } \xi \in R^n; \tag{3.2}$$

(F2$^*$) the functions $b_i = \partial F/\partial u_i$, $c = \partial F/\partial u$ satisfy

$$|b_i| \leq K, \quad -K \leq c \leq 0. \tag{3.3}$$

We notice that our equations (3.1) include the Bellman equations (1.1), if $a_{ij} = a_{ij}^m$ satisfy (3.2), $c^m \le 0$ for all $m$, and the norms in $C^{\alpha}(\Omega)$,

$$|a_{ij}^m,\, b_i^m\, c^m|_{\alpha;\Omega} \le K, \quad |f^m|_{\alpha;\Omega} \le K_1.$$

Consequently, the following Theorems 3.1–3.3 can be viewed as generalizations of known Schauder-type results for linear equations (see [9], Theorems 6.13, 6.14, 6.31; [19], Ch.3).

**Theorem 3.1.** *Let $\Omega$ be a bounded Lipschitz (satisfying Assumptions 2.1) domain in $R^n, d_0 = diam\,\Omega \le R_0 = const < \infty$, $\varphi \in C(\overline{\Omega})$, and let $F(x, u, u_i, u_{ij})$ satisfy Assumptions 3.1. Then the Dirichlet problem*

$$F[u] = 0 \text{ in } \Omega, \quad u = \varphi \text{ on } \partial\Omega \tag{3.4}$$

*has a unique solution $u \in C^{2,\alpha;0}(\Omega) \cap C(\overline{\Omega})$, provided $0 < \alpha < \overline{\alpha}$ for some constant $\overline{\alpha} = \overline{\alpha}(n, \nu) \in (0, 1)$. Moreover, we have*

$$U_0 = \sup_{\Omega} |u| \le \sup_{\partial\Omega} |\varphi| + N(n, \nu, K, d_0) \cdot K_1, \tag{3.5}$$

$$\|u\|_{2,\alpha;\Omega}^{(0)} \le N(n, \nu, K, \alpha, R_0) \cdot (U_0 + d_0^{2+\alpha} K_1). \tag{3.6}$$

We will use the following definition for the classification of the boundaries $\partial\Omega$ having higher then the Lipschitz smoothness.

**Definition 3.1.** The boundary $\partial\Omega$ of a bounded domain $\Omega \subset R^n$ belongs to the class $C^{k,\alpha}$ for $k = 1, 2, \cdots, 0 < \alpha \le 1$, if there exists a function $\Psi(x) \in C^{k,\alpha}(R^n)$ such that

$$\Omega = \{x \in R^n : \Psi(x) > 0\} \quad \text{and} \quad |D\Psi| \ge 1 \text{ on } \partial\Omega.$$

**Theorem 3.2.** *Let $\Omega$ be a bounded domain in $R^n$ with the boundary $\partial\Omega \in C^{2,\alpha}$, $\varphi \in C^{2,\alpha}(\overline{\Omega})$, and let $F(x, u, u_i, u_{ij})$ satisfy Assumptions 3.1. Then the Dirichlet problem (3.4) has a unique solution $u \in C^{2,\alpha}(\overline{\Omega})$, provided $0 < \alpha < \overline{\alpha}$ for some constant $\overline{\alpha} = \overline{\alpha}(n, \nu) \in (0, 1)$. Moreover, we have (3.5) and*

$$|u|_{2,\alpha;\Omega} \le N(n, \nu, K, \alpha, \Omega) \cdot (U_0 + K_1 + |\varphi|_{2,\alpha;\Omega}). \tag{3.7}$$

**Theorem 3.3.** *Let $\Omega$ be a bounded domain in $R^n$ with the boundary $\partial\Omega \in C^{1,\alpha}$, the functions $b_0, b_1, \cdots, b_n, g \in C^{1,\alpha}(\overline{\Omega})$, and let $F(x, u, u_i, u_{ij})$ satisfy Assumptions 3.1. Suppose that for some constant $\nu_0 > 0$,*

$$b_0 \ge \nu_0, \ \ b \cdot \mathcal{N} = \sum_i^n b_i \mathcal{N}_i \ge \nu_0 |b| > 0 \text{ on } \partial\Omega, \tag{3.8}$$

*where $\mathcal{N} = -|D\Psi|^{-1} D\Psi$ is the outward unit normal on $\partial\Omega$. Then the oblique derivative problem*

$$F[u] = 0 \text{ in } \Omega, \quad Bu = b_0 u + b \cdot Du = g \text{ on } \partial\Omega \tag{3.9}$$

*has a unique solution* $u \in C^{2,\alpha}(\overline{\Omega})$*, provided* $0 < \alpha < \overline{\alpha}$ *for some constant* $\overline{\alpha} = \overline{\alpha}(n, \nu, \nu_0) \in (0,1)$*. Moreover, we have*

$$(3.10) \qquad |u|_{2,\alpha;\Omega} \le N \cdot (U_0 + K_1 + |g|_{1,\alpha;\Omega}),$$

*where the constant* $N$ *depends only on* $n, \nu, K, \nu_0$*, domain* $\Omega$ *(with* $\partial\Omega \in C^{1,\alpha}$*), and on the norms of the functions* $b_0, b_1, \cdots, b_n$ *in* $C^{1,\alpha}(\overline{\Omega})$*.*

Theorems 3.1, 3.2 are similar to Theorems 1.1, 1.2 from [26], Theorem 3.3 for Bellman equations (1.1) in $\Omega$ with $\partial\Omega \in C^{2,\alpha}$ was proved in [1]. Analogous results are also true in the parabolic case, with the same modifications as in the theory of linear equations ([18], Theorems 5.1–5.3 in Ch.4).

Below, in Sections 4–8, we describe the technique of deriving $C^{2,\alpha}$-estimates for solutions of the problems (3.4), (3.9) by given constants $n, K, K_1, \alpha, \cdots$. On the grounds of the $C^{2,\alpha}$-estimates, one can prove Theorems 3.1–3.3 by the standard continuity method (see [9], Ch.17; [13], Sec. 1.3). Therefore, we will only supplement these sections with some remarks concerning the continuity method applied to our problems.

Our approach to $C^{2,\alpha}$-estimates in general case uses analogous estimates in the case of simplest nonlinear equations and the comparison principle for nonlinear equations. The simplest nonlinear equations are defined as follows.

**Definition 3.2.** The *simplest* nonlinear elliptic equation has the form $F_0[u] = F_0(D^2u) = 0$, where the function $F_0(u_{ij})$ satisfies (F0), (F1) on $S^n$ with some constant $\nu \in (0,1]$, and $F_0(0) = 0$. We denote $\mathcal{F}(\nu)$ the class of all such functions $F_0(u_{ij})$.

The comparison principle is based on the next simple lemma.

**Lemma 3.1.** *Let* $F(x, u, u_i, u_{ij})$ *satisfies (F0)–(F2) with some constants* $K \ge 0, \nu \in (0,1]$*, and let* $u, v \in C^2(\Omega)$*. Then* $F[u] - F[v] = L(u - v)$ *in* $\Omega$*, where the linear elliptic operator* $L = a_{ij}D_{ij} + b_iD_i + c$ *has the coefficients satisfying (3.2), (3.3) on* $\Omega$*. Furthermore, in* $F_0(u_{ij}) \in \mathcal{F}(\nu)$*, then*

$$(3.11) \qquad F_0[u] - F_0[v] = L_0(u - v) = a_{ij}D_{ij}(u - v) \quad \text{in } \Omega,$$

*where* $a_{ij} = a_{ij}(x)$ *satisfy (3.2) on* $\Omega$*. In particular (*$v = 0$*),*

$$(3.12) \qquad F_0[u] = L_0u = a_{ij}D_{ij}u \quad \text{in } \Omega.$$

The coefficients $a_{ij}, b_i, c$ can be constructed directly (see [26], Lemma 1.1) or through approximation of $F$ by smooth functions. For example, (3.11) follows from the equality

$$(3.13) \qquad F_0(u_{ij}) - F_0(v_{ij}) = \overline{a}_{ij} \cdot (u_{ij} - v_{ij}) \quad \text{for all } [u_{ij}], [v_{ij}] \in S^n,$$

where $\overline{a}_{ij}$ (depending on $u_{ij}, v_{ij}$) satisfy (3.2). If $F_0(u_{ij}) \in \mathcal{F}(\nu)$ is smooth, then we can take in (3.13)

$$\overline{a}_{ij} = \int_0^1 a_{ij}(\theta u_{ij} + (1-\theta)v_{ij})\, d\theta, \quad \text{where } a_{ij}(u_{ij}) = \partial F_0(u_{ij})/\partial u_{ij},$$

and (3.11) holds by virtue of (F1$^*$ ).

From this lemma and the classical maximum principle (see [9], Theorem 3.7; [19], Ch.3, §1), we get

**Corollary 3.1.** *Let $F$ satisfy (F0)–(F2), and let $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$. Then*

$$\sup_{\Omega} |u - v| \leq \sup_{\partial\Omega} |u - v| + N(n, \nu, K, diam\, \Omega) \cdot \sup_{\Omega} |F[u] - F[v]|.$$

*In particular, under the assumptions of Theorems 3.1 and 3.2, there exists at most one solution of the problem (3.4).*

Taking in this corollary $v = 0$, we arrive at the following:

**Corollary 3.2.** *Under the assumptions of Theorems 3.1 and 3.2, the estimate (3.5) holds.*

**Corollary 3.3.** *If $u \in C^2(\overline{\Omega})$ is a solution of the simplest nonlinear equaiton $F_0[u] = 0$ in $\Omega$, then $\sup_{\Omega} |u| = \sup_{\partial\Omega} |u|$.*

## 4. Interior $C^{2,\alpha}-$ estimates: the simplest nonlinear equations

In this section we will obtain the interior $C^{2,\alpha}$-estimates for solutions of simplest nonlinear equations $F_0[u] = F_0(D^2u) = 0$. In the next section, these estimates will be applied to the proof of similar estimates for solutions of general nonlinear equations.

**Theorem 4.1.** *Let $\nu \in (0, 1], x_0 \in R^n, r > 0, B_r = B_r(x_0), \varphi \in C(\overline{B_r})$ , and the function $F_0(u_{ij}) \in \mathcal{F}(\nu)$. Then the problem*

$$F_0[v] = F_0(D_{ij}v) = 0 \text{ in } B_r, \quad v = \varphi \text{ on } \partial B_r \tag{4.1}$$

*has a unique solution $v \in C^{2,\overline{\alpha};0}(B_r) \cap C(\overline{B_r})$ , and*

$$\|v\|^{(0)}_{2,\overline{\alpha};B_r} \leq N \cdot \sup_{\partial B_r} |\varphi|, \tag{4.2}$$

*where the constants $\overline{\alpha} \in (0, 1]$ and $N > 0$ depend only on $n, \nu$.*

This theorem (for more general equations) was proved independently by N.V. Krylov [11] and L.C. Evans [7] (see also [9], Section 17.4). We give here another proof which we precede with three lemmas. We will use the following lemma from [14], [22] (see also [13], [9], Sec. 9.8).

**Lemma 4.1.** *Let $\nu \in (0, 1], \rho > 0, \mu \in (0, 1], V \in C^2(B_{2\rho})$, and suppose that $V \geq 0$, $a_{ij}D_{ij}V \leq 0$ in $B_{2\rho}$, where the functions $a_{ij} = a_{ij}(x)$ satisfy*

$$a_{ij} = a_{ji}, \quad \nu|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \nu^{-1}|\xi|^2 \text{ for all } \xi \in R^n. \tag{4.3}$$

*Moreover, let the Lebesgue measure $|\{x \in B_\rho : V(x) \geq 1\}| \geq \mu \cdot |B_\rho|$. Then*

$$\inf_{B_\rho} V \geq \beta = \beta(n, \nu, \mu) > 0.$$

The next lemma is standard in the study of Hölder spaces.

**Lemma 4.2.** *Let constants* $q > 1, \alpha > 0, \rho_0 > 0$ *be given, and let* $\omega(\rho)$ *be a positive non-decreasing function on* $(0, \rho_0]$ *satisfying the inequality*

$$q^\alpha \omega(\rho) \le \omega(q\rho) \quad \text{for all} \quad \rho \in (0, \rho_0/q). \tag{4.4}$$

*Then*

$$\rho^{-\alpha}\omega(\rho) \le q^\alpha \rho_0^{-\alpha}\omega(\rho_0) \quad \text{for all} \quad \rho \in (0, \rho_0]. \tag{4.5}$$

*Proof.* By virtue of monotony of $\omega(\rho)$, (4.5) is evident for $\rho \in (\rho_0/q, \rho_0]$. If $\rho < \rho_0/q$, then $q^k\rho \in (\rho_0/q, \rho_0]$ for some natural $k$, and using (4.4), we obtain:

$$\rho^{-\alpha}\omega(\rho) \le (q\rho)^{-\alpha}\omega(q\rho) \le \cdots \le (q^k\rho)^{-\alpha}\omega(q^k\rho) \le q^\alpha\rho_0^{-\alpha}\omega(\rho_0).$$

□

**Lemma 4.3.** *For* $v \in C^2(B_\rho)$, *let us set*

$$\omega = \omega(\rho) = \max_{i,j} \operatorname*{osc}_{B_\rho} D_{ij}v, \tag{4.6}$$

*and*

$$\omega^* = \omega^*(\rho) = \int_\Lambda \operatorname*{osc}_{B_\rho} D_{\lambda\lambda}v \, ds_\lambda,$$

*where the surface integral over* $\Lambda = \partial B_1(0)$ *is considered. Then*

$$N^{-1}\omega \le \omega^* \le N\,\omega, \quad \text{where} \;\; N = N(n). \tag{4.7}$$

*Proof.* Consider the function $Q(\lambda, x) = D_{\lambda\lambda}v(x) = \lambda_i\lambda_j D_{ij}v(x)$ on the set $R^n \times B_\rho$. Notice that for all $i, j, \lambda, x$,

$$2\,D_{ij}v(x) = Q(\lambda + e_i + e_j, x) - Q(\lambda + e_i, x) - Q(\lambda + e_j, x) + Q(\lambda, x).$$

Integrating this inequality over $\lambda \in B_1(0)$, we obtain:

$$2\,|B_1| \cdot D_{ij}v(x) = \left\{ \int_{B_1(e_i+e_j)} - \int_{B_1(e_i)} - \int_{B_1(e_j)} + \int_{B_1(0)} \right\} Q(\lambda, x)\, d\lambda.$$

In the right-hand side we have four integrals over unit balls with centers at $e_i + e_j, e_i, e_j, 0 \in R^n$. All these balls are contained in $B_3(0)$, therefore,

$$2\,|B_1| \cdot \omega \le 4 \int_{B_3(0)} \operatorname*{osc}_{x \in B_\rho} Q(\lambda.x)\, d\lambda. \tag{4.8}$$

Further, $Q(r\lambda, x) \equiv r^2 Q(\lambda, x)$ for all $r > 0$. Hence by passing to the spherical coordinates, the integral in (4.8) can be rewritten as follows:

$$\int_0^3 dr \int_{\partial B_r(0)} \operatorname*{osc}_{x \in B_\rho} Q(\lambda, x)\, ds_\lambda = \int_0^3 r^{n+1} dr \int_\Lambda \operatorname*{osc}_{x \in B_\rho} D_{\lambda\lambda}v \cdot ds_\lambda = \frac{3^{n+2}}{n+2}\omega^*.$$

This relation together with (4.8) give us the first inequality in (4.7). The second inequality is obvious because for all $\lambda \in \Lambda$,

$$\operatorname*{osc}_{x\in B_\rho} D_{\lambda\lambda} v \le \omega \cdot \sum_{i,j} |\lambda_i \lambda_j| \le \frac{\omega}{2} \sum_{i,j} (\lambda_i^2 + \lambda_j^2) = n\omega.$$

$\square$

*Proof of Theorem 4.1.* Now we will prove only the estimate (4.2) assuming that the problem (4.1) is solvable in $C^{2,\overline{\alpha};0}(B_r) \cap C(\overline{B_r})$. This assumption will be substantiated below, in Remark 5.1.

*Step 1.* Notice that each function $F_0(u_{ij}) \in \mathcal{F}(\nu)$ can be easy approximated by smooth functions $F_0^\delta(u_{ij}) \in \mathcal{F}(\nu)$, $\delta > 0$, such that

$$(4.9) \qquad |F_0(u_{ij}) - F_0^\delta(u_{ij})| \le \delta \quad \text{for all } [u_{ij}] \in S^n, \ \delta > 0.$$

If $v^\delta$ is the solution of the problem

$$F_0^\delta[v^\delta] = 0 \text{ in } B_r, \quad v^\delta = \varphi \text{ on } \partial B_r,$$

then by Corollary 3.1 and (4.9) we obtain

$$\sup_{B_r} |v^\delta - v| \le N \cdot \sup_{B_r} |F_0[v^\delta] - F_0[v]| \le N \cdot \sup_{B_r} |F_0[v^\delta] - F_0^\delta[v^\delta]| \le N\delta.$$

Therefore, if $v^\delta$ satisfy the estimate (4.2) for all $\delta > 0$, then this estimate remains valid for $v$, the solution of initial problem (4.1). Thus we can assume without loss of generality that $F_0(u_{ij})$ is smooth. Then automatically $v \in C^\infty(B_r)$ (see [13], Lemma I.3.2; [9], Lemma 17.16). Next, replacing $r$ by $r - \varepsilon$ and then letting $\varepsilon \to 0+$, we can assume $v \in C^4(\overline{B_r})$.

*Step 2.* Let us fix

$$(4.10) \qquad z \in B_r, \ \ 0 < \rho \le d(z) = \frac{1}{2}\operatorname{dist}(z, \partial B_r), \ \ \omega = \omega(\rho) = \max_{i,j} \operatorname*{osc}_{B_\rho(z)} D_{ij} v.$$

Relying on the smoothness of $F_0(u_{ij})$ and $v(x)$, we differentiate the equality $F_0[u] = 0$ twice in the direction $\lambda \in \Lambda$. Using (F0*), (F1*) in Remark 3.1, we have

$$(4.11) \qquad a_{ij} D_{ij} D_{\lambda\lambda} v = -\partial^2 F_0 / \partial u_{ij} \partial u_{pq} \cdot D_{ij} D_\lambda v \cdot D_{pq} D_\lambda v \le 0 \quad \text{in } B_r,$$

where $a_{ij} = \partial F_0 / \partial u_{ij}$. Hence the functions

$$V^\lambda(x) = D_{\lambda\lambda} v(x) - \inf_{B_{2\rho}(z)} D_{\lambda\lambda} v$$

satisfy the relations

$$(4.12) \qquad V^\lambda \ge 0, \ \ a_{ij} D_{ij} V^\lambda \le 0 \quad \text{in } B_{2\rho}(z).$$

*Step 3.* Next, we fix $x \in B_\rho(z)$. From the definition of $\omega$ in (4.10) it follows that there exists a point $y \in \overline{B_\rho(z)}$ such that

$$\omega = \omega(\rho) \ge \max_{i,j} |D_{ij} u(x) - D_{ij} u(y)| \ge \omega/2.$$

By virtue of (3.13), we have $\overline{a}_{ij}\cdot(D_{ij}v(x)-D_{ij}v(y))=0$, where $\overline{a}_{ij}$ (depending on $x,y$) satisfy (4.3). Therefore, if $\mu_1\le\mu_2\le\cdots\le\mu_n$ are eigenvalues of the matrix $M=[M_{ij}]=[D_{ij}v(x)-D_{ij}v(y)]$, then $\mu_n\ge 2\mu\omega$ for some constant $\mu=\mu(n,\nu)>0$. Moreover, we can assume $\mu=\mu(n,\nu)>0$ to be suitable chosen so that the Lebesgue measure on $\Lambda=\partial B_1(0)$,

$$|\{\lambda\in\Lambda: M_{ij}\lambda_i\lambda_j\ge\mu\omega\}|\ge 2\mu\cdot|\Lambda|.$$

Since $x\in B_\rho(z)$ can be taken arbitrarily, and

$$M_{ij}\lambda_i\lambda_j=D_{\lambda\lambda}u(x)-D_{\lambda\lambda}u(y)\le V^\lambda(x),$$

we arrive at the estimate

$$|\{\lambda\in\Lambda: V^\lambda(x)\ge\mu\omega\}|\ge 2\mu\cdot|\Lambda|\quad\text{for all } x\in B_\rho(z). \tag{4.13}$$

*Step 4.* Now we set

$$\Gamma=\{(\lambda,x)\in\Lambda\times B_\rho(z): V^\lambda(x)\ge\mu\omega\}\subset\Lambda\times B_\rho(z),$$

$$\Gamma_\lambda=\{x\in B_\rho(z): V^\lambda(x)\ge\mu\omega\}\quad\text{for }\lambda\in\Lambda,$$

$$\Gamma_x=\{\lambda\in\Lambda: V^\lambda(x)\ge\mu\omega\}\quad\text{for }x\in B_\rho(z),$$

$$\Lambda_0=\{\lambda\in\Lambda: |\Gamma_\lambda|\ge\mu\cdot|B_\rho|\}. \tag{4.14}$$

By (4.13) and Fubini's theorem, the product-measure on $\Lambda\times B_\rho(z)$,

$$|\Gamma|=\int_{B_\rho(z)}|\Gamma_x|\,dx\ge 2\mu\cdot|\Lambda|\cdot|B_\rho|.$$

On the other hand, by definition of $\Lambda_0$,

$$\int_{\Lambda\setminus\Lambda_0}|\Gamma_\lambda|\,ds_\lambda\le|\Lambda\setminus\Lambda_0|\cdot\mu|B_\rho|\le\mu|\Lambda|\cdot|B_\rho|,$$

therefore,

$$|\Lambda_0|\cdot|B_\rho|\ge\int_{\Lambda_0}|\Gamma_\lambda|\,ds_\lambda=|\Gamma|-\int_{\Lambda\setminus\Lambda_0}|\Gamma_\lambda|\,ds_\lambda\ge\mu\cdot|\Lambda|\cdot|B_\rho|,$$

so we obtain $|\Lambda_0|\ge\mu\cdot|\Lambda|$.

*Step 5.* The relations (4.12) allow us to apply Lemma 4.1 to the functions $V(x)=(\mu\omega)^{-1}V^\lambda(x),\quad\lambda\in\Lambda_0$. This gives us

$$\operatorname*{osc}_{B_{2\rho}(z)}D_{\lambda\lambda}v-\operatorname*{osc}_{B_\rho(z)}D_{\lambda\lambda}v\ge\inf_{B_\rho(z)}D_{\lambda\lambda}v-\inf_{B_{2\rho}(z)}D_{\lambda\lambda}v=\inf_{B_\rho(z)}V^\lambda\ge\beta\mu\cdot\omega(\rho)$$

for all $\lambda\in\Lambda_0$. Taking into account (4.14) and using Lemma 4.3, we obtain:

$$\omega^*(2\rho)-\omega^*(\rho)=\int_\Lambda\left[\operatorname*{osc}_{B_{2\rho}(z)}D_{\lambda\lambda}v-\operatorname*{osc}_{B_\rho(z)}D_{\lambda\lambda}v\right]ds_\lambda\ge|\Lambda_0|\cdot\beta\mu\cdot\omega(\rho)\ge\mu_1\cdot\omega^*(\rho)$$

for some constant $\mu_1 = \mu_1(n, \nu) > 0$.

*Step 6.* Taking $\overline{\alpha} = \overline{\alpha}(n, \nu) = \log_2(1 + \mu_1) > 0$ and $\rho_0 = d(z)$, we see that

$$2^{\overline{\alpha}} \omega^*(\rho) = (1 + \mu_1)\omega^*(\rho) \le \omega^*(2\rho) \quad \text{for all} \quad \rho \in (0, \rho_0/2].$$

Using Lemma 4.2 with $q = 2$ and then again Lemma 4.3, we get

$$\rho^{\overline{\alpha}} \omega(\rho) \le N(n, \nu)\rho_0^{-\overline{\alpha}} \omega(\rho_0) \quad \text{for all} \quad \rho \in (0, \rho_0]. \tag{4.15}$$

This estimate implies

$$\rho_0^{2+\overline{\alpha}} \rho^{-\overline{\alpha}} \omega(\rho) \le N \rho_0^2 \omega(\rho_0) \le N \cdot [v]^{(0)}_{2,0;B_r}.$$

Taking the *sup* over $z \in B_r$ and $\rho \in (0, \rho_0] = (0, d(z)]$, and applying Theorem 2.1, we get

$$[v]^{(0)}_{2,\overline{\alpha}} \le N\, \hat{M}^{(0)}_{2,\overline{\alpha}} \le N\, [v]^{(0)}_{2,0}.$$

Finally, from the interpolation inequalities (2.12) and Corollary 3.3 it follows

$$\|v\|^{(0)}_{2,\overline{\alpha}} \le N \sup_{\Omega} |v| \le N \sup_{\partial\Omega} |\varphi|.$$

So we have proved the desired estimate (4.2). □

**Remark 4.1.** Theorem 4.1 is the only point where the convexity condition (F0) is used. It is an *open problem* whether it remains valid without (F0). In the case $n = 2$, this is so and follows from well-known result of Nirenberg [21] on $C^{1,\alpha}$-smoothness of solutions of linear elliptic equations with measurable coefficients. Therefore, in the case $n = 2$ all our results, including Theorems 3.1–3.3, remain in effect without (F0). It was shown in [27] that the result of Nirenberg fails for $n = 3$, and hence the convexity condition (F0) is essential in our considerations when $n \ge 3$.

## 5. Interior $C^{2,\alpha}-$ estimates: general equations

We will use the interior estimate (4.2) in $C^{2,\overline{\alpha};0}$ for solutions of simplest nonlinear equations in the proof of estimates in $C^{2,\alpha;0}$, $0 < \alpha < \overline{\alpha}$ for solutions of general nonlinear equations. In the case of linear equations, we can consider linear equations $L_0 v = \sum a_{ij} D_{ij} v = 0$ with constant coefficients in place of simplest nonlinear equations $F_0[v] = 0$, so (4.2) is true with $\overline{\alpha} = 1$, and hence for general linear equations with coefficients in $C^\alpha$, we can take arbitrary $\alpha \in (0, 1)$.

**Theorem 5.1.** *Let $\Omega$ be a bounded domain in $R^n$ with $d_0 = diam\, \Omega \le R_0 = const < \infty$, and let $F(x, u, u_i, u_{ij})$ satisfy Assumptions 3.1 with some constants $K, K_1 \ge 0, \nu \in (0, 1], \alpha \in (0, \overline{\alpha})$, where $\overline{\alpha} = \overline{\alpha}(n, \nu)$ is the constant in Theorem 4.1. Suppose $u \in C^{2,\alpha;0}(\Omega) \cap C(\overline{\Omega})$ is a solution of the equation $F[u] = 0$ in $\Omega$. Then the estimate (3.6) holds.*

*Proof.* Throughout the proof we will denote by $N$ different constants depending only on $n, \nu, K, \alpha, R_0$. We will also use the brief notations

$$U_{2,\alpha} = [u]^{(0)}_{2,\alpha}, \quad U_k = [u]^{(0)}_{k,0}.$$

*Step 1.* Let us fix

$$y \in \Omega, \quad d = d(y) = \frac{1}{2}\operatorname{dist}(y, \partial\Omega), \quad \rho \in (0, d], \quad \text{and} \quad \varepsilon \in (0, 1/2].$$

We set $r = \rho/\varepsilon$ and consider separately the cases (a) $r \le d$ and (b) $r > d$.

In the case (a), we take

$$\varphi = u - T_{y,2}u,\ \overline{u} = u(y),\ \overline{u}_i = D_i u(y),\ \overline{u}_{ij} = D_{ij}u(y),$$

and $F_0(u_{ij}) = F(y, \overline{u}, \overline{u}_i, \overline{u}_{ij} + u_{ij})$. Since $F_0(0) = F[u](y) = 0$, $F_0 \in \mathcal{F}(\nu)$. Next, we determine $v$ as the solution of the problem

$$F_0[v] = F_0(D_{ij}v) = 0 \ \text{ in } \ B_r = B_r(y), \quad v = \varphi \ \text{ on } \ \partial B_r. \tag{5.1}$$

From Theorem 4.1 it follows $v \in C^{2,\overline{\alpha};0}(B_r) \cap C(\overline{B_r})$ and

$$r^{2+\overline{\alpha}}[v]_{2,\overline{\alpha};B_{r/2}} \le N \cdot \sup_{\partial B_r} |\varphi|.$$

Having in mind that $\rho = \varepsilon r \le r/2$, applying Corollary 2.2 to $v$ in $B_\rho$, and then Lemma 2.2 to $u$ in $B_r$, we obtain (with different constants $N$):

$$\rho^{-2-\alpha}E_2[v; B_\rho] \le N\rho^{\overline{\alpha}-\alpha}[v]_{2,\overline{\alpha};B_{r/2}} \le N\rho^{\overline{\alpha}-\alpha}r^{-2-\overline{\alpha}} \sup_{B_r}|\varphi|$$

$$\le N\rho^{\overline{\alpha}-\alpha}r^{\alpha-\overline{\alpha}}[u]_{2,\alpha;B_r} = N\varepsilon^{\overline{\alpha}-\alpha}[u]_{2,\alpha;B_r}.$$

Since $r \le d$, by definition of $U_{2,\alpha} = [u]^{(0)}_{2,\alpha}$ in (2.8), we get

$$d^{2+\alpha}\rho^{-2-\alpha}E_2[v; B_\rho] \le N\varepsilon^{\overline{\alpha}-\alpha}U_{2,\alpha}. \tag{5.2}$$

*Step 2.* Further,we will evaluate $F_0[\varphi]$ on $B_r = B_r(y)$. For $x \in B_r$, we set $u = u(x), u_i = D_i u(x), u_{ij} = D_{ij}u(x)$. Notice that $D_{ij}\varphi(x) = u_{ij} - \overline{u}_{ij}$. Therefore,

$$|F_0[\varphi](x)| = |F_0[\varphi](x) - F[u](x)| = |F(y, \overline{u}, \overline{u}_i, u_{ij}) - F(x, u, u_i, u_{ij})|$$

$$\le |F(y, \overline{u}, \overline{u}_i, u_{ij}) - F(y, u, u_i, u_{ij})| + |F(y, u, u_i, u_{ij}) - F(x, u, u_i, u_{ij})|$$

in $B_r$. Since $|x - y| < r \le d \le d_0 \le R_0$, we have

$$|u - \overline{u}| + \sum_i |u_i - \overline{u}_i| \le Nr \cdot |u|_{2,0;B_r} \le Nr^\alpha \cdot |u|_{2,0;B_r}.$$

By virtue of (F2) and (F4),

$$A = r^{-\alpha}\sup_{B_r}|F_0[\varphi]| \le N \cdot \left(|u|_{2,0;B_r} + K_1\right),$$

hence

$$d^{2+\alpha}A \le N\cdot(U_2+U_1+U_0+d_0^{2+\alpha}K_1). \tag{5.3}$$

Now we proceed to evaluate $\varphi - v$ on $B_r$. By Lemma 3.1,

$$F_0[\varphi] = F_0[\varphi] - F_0[v] = L_0(\varphi - v) = a_{ij}D_{ij}(\varphi - v) \quad \text{in} \quad B_r$$

with $a_{ij}$ satisfying (3.2). The functions $\varphi - v$ together with

$$w(x) = \frac{Ar^\alpha}{2n\nu}(r^2 - |x-y|^2)$$

satisfy the relations

$$L_0 w \le -Ar^\alpha \le -|L_0(\varphi - v)| \;\text{ in }\; B_r = B_r(y), \quad w = \varphi - v = 0 \;\text{ on }\; \partial B_r.$$

By the comparison principle, we get

$$\sup_{B_\rho}|\varphi - v| \le \sup_{B_r}|\varphi - v| \le \sup_{B_r}|w| = \frac{A}{2n\nu}r^{2+\alpha}.$$

Using the equality $r = \rho/\varepsilon$ and (5.3), we obtain the estimate

$$d^{2+\alpha}\rho^{-2-\alpha}\sup_{B_\rho}|\varphi - v| \le N\varepsilon^{-2-\alpha}(U_2+U_1+U_0+d_0^{2+\alpha}K_1). \tag{5.4}$$

*Step 3.* Now we will combine together (5.2) and (5.4). Obviously

$$E_2[u;B_\rho] \le E_2[v;B_\rho] + E_2[\varphi - v;B_\rho] \le E_2[v;B_\rho] + \sup_{B_\rho}|\varphi - v|,$$

so we receive

$$d^{2+\alpha}\rho^{-2-\alpha}E_2[u;B_\rho] \le N\varepsilon^{\overline{\alpha}-\alpha}U_{2,\alpha} + N\varepsilon^{-2-\alpha}(U_2+U_1+U_0+d_0^{2+\alpha}K_1). \tag{5.5}$$

We have considered the case (a) $r = \rho/\varepsilon \le d$. In the case (b) $r = \rho/\varepsilon > d$, we have $d^{2+\alpha}\rho^{-2-\alpha} < \varepsilon^{-2-\alpha}$, and $E_2[u;B_\rho] \le \sup_{B_d}|u| \le U_0$, so the left hand side of (5.5) does not exceed $\varepsilon^{-2-\alpha}U_0$. Since $y \in \Omega$ and $0 < \rho \le d = d(y)$ are chosen in an arbitrary manner, we get the following estimate for the seminorm in Theorem 2.1:

$$M^{(0)}_{2,\alpha} \le N\varepsilon^{\overline{\alpha}-\alpha}U_{2,\alpha} + N\varepsilon^{-2-\alpha}(U_2+U_1+U_0+d_0^{2+\alpha}K_1) \tag{5.6}$$

for all $\varepsilon > 0$. By this theorem, (5.6) remains valid with $U_{2,\alpha}$ in place of $M^{(0)}_{2,\alpha}$. Choosing then $\varepsilon = \varepsilon(n,\nu,K,\alpha,R_0) > 0$ such that the coefficient of $U_{2,\alpha}$ would be less then $1/2$, we get

$$U_{2,\alpha} \le N\cdot(U_2+U_1+U_0+d_0^{2+\alpha}K_1). \tag{5.7}$$

Finally, from (5.7) and the interpolation inequalities (2.12),

$$U_2 + U_1 \le \varepsilon\, U_{2,\alpha} + N(\varepsilon)U_0, \quad \varepsilon > 0,$$

it follows

$$\|u\|^{(0)}_{2,\alpha} = U_{2,\alpha} + U_2 + U_1 + U_0 \le N\cdot(U_0 + d_0^{2+\alpha}K_1),$$

completing the proof of theorem. □

**Remark 5.1.** We relied on Theorem 4.1, though its proof given in Section 4 was not quite complete because we assumed the solvability of the problem (4.1). This gap can be removed with the help of some variant of the method of continuation with respect to the parameter $\nu$. By virtue of (3.12), in the case $\nu = 1$ the class $\mathcal{F}(\nu) = \mathcal{F}(1)$ consists of the only function $tr[u_{ij}] = \sum u_{ii}$ corresponding to the Laplace operator $\Delta$, so the standard results of the linear theory of elliptic equations yield all the statements of Theorem 4.1 with $\overline{\alpha} = 1$.

Setting out from $\nu' = 1$, $F'(u_{ij}) = tr[u_{ij}]$, and using (3.12), we notice that the functions $F_0(u_{ij}) \in \mathcal{F}(\nu)$, $0 < \nu < 1$, satisfy the estimate

$$|F_0'(u_{ij}) - F_0(u_{ij})| \le \gamma \cdot \max_{i,j} |u_{ij}|, \tag{5.8}$$

where the constant $\gamma = \gamma(n, \nu) \to 0$ as $\nu \to 0$. In the proof of Theorem 5.1 with $\nu$ close to $\nu' = 1$, one can consider the solution $v$ of the problem

$$F_0'[v] = F_0(D_{ij}v) = 0 \text{ in } B_r = B_r(y), \quad v = \varphi \text{ on } \partial B_r$$

instead of the problem (5.1). Then the estimate (5.2) remains valid. By virtue of (5.8) we have

$$\sup_{B_r} |F_0'[\varphi] - F_0[\varphi]| \le \gamma[\varphi]_{2,0;B_r} \le \gamma r^{\alpha}[\varphi]_{2,\alpha;B_r},$$

$$A' = r^{-\alpha} \sup_{B_r} |F_0'[\varphi]| \le \gamma[\varphi]_{2,\alpha;B_r} + A.$$

The last estimate together with (5.3) imply

$$d^{2+\alpha}A' \le \gamma U_{2,\alpha} + N \cdot (U_2 + U_1 + U_0 + d_0^{2+\alpha}K_1).$$

Hence in the right hand sides of (5.4)–(5.6) we will have an additional term $N\varepsilon^{-2-\alpha}\gamma U_{2,\alpha}$. If $\nu$ is close enough to 1, then $\gamma$ is small and the coefficient of $U_{2,\alpha}$ in (5.6) still can be made less then 1/2.

Thus, starting from $\nu' = 1$, we see that Theorem 5.1 remains true for some $\nu = \nu_1 < 1$. As was pointed out in Section 3, Theorem 3.1 can be obtained on the grounds of the estimates provided by Corollary 3.2 and Theorem 5.1. In turn, Theorem 4.1 is a special case of Theorem 3.1 with $\Omega = B_r$ and $F = F_0$. So, all Theorems 3.1, 4.1, and 5.1 are true for $\nu = \nu_1$.

Further moving past $\nu' = \nu_1$, we can approximate $F_0(u_{ij}) \in \mathcal{F}(\nu)$ by the functions

$$F_0'(u_{ij}) = \theta\delta_{ij} + (1-\theta)F_0(u_{ij}), \quad 0 < \theta < 1.$$

Moreover, if $\nu < \nu_1$ is close to $\nu_1$, then for small $\theta > 0$ we have $F_0'(u_{ij}) \in \mathcal{F}(\nu_1)$ and the constant $\gamma$ in (5.8) will also be small. Hence all the previous considerations are valid for some $\nu = \nu_2 < \nu_1$. Continuing this procedure, we can embrace the arbitrary $\nu \in (0, 1]$.

## 6. Some boundary estimates for solutions of linear elliptic equations

We will essentially use the following Lemma 6.1 announced in [23] (see also [25]). For applications to nonlinear equations, the same results can be obtained by Krylov's method (see [12] and [13], comments to §1 of Ch.5), which implies the consideration of auxiliary degenerate elliptic or parabolic equation for $V(x)/x_n$.

**Lemma 6.1.** *Let* $\nu \in (0,1]$, $x_0 \in R^n_0$, $r > 0$, *and* $B^+_r = R^n_+ \cap B_r(x_0)$. *Suppose that* $V \in C^2(B^+_r) \cap C(\overline{B^+_r})$ *be a solution of the equation*

$$a_{ij} D_{ij} V = 0 \quad \text{in } B^+_r, \tag{6.1}$$

*where* $a_{ij} = a_{ij}(x)$ *satisfy the conditions (4.3), and moreover,*

$$V = 0 \quad \text{on } \Gamma = R^n_0 \cap B_r(x_0). \tag{6.2}$$

*Then the function* $\omega(\rho) = \operatorname*{osc}_{B^+_\rho} V(x)/x_n$ *satisfies the estimate*

$$\rho^{-\overline{\alpha}}\omega(\rho) \le N r^{-\overline{\alpha}}\omega(r) \quad \text{for all} \quad \rho \in (0,r], \tag{6.3}$$

*where the constants* $\overline{\alpha} \in (0,1]$, $N > 0$ *depend only on* $n, \nu$.

*Proof.* By Lemma 4.2, for the proof of (6.3) it is sufficient to obtain, for example, the estimate

$$10^{\overline{\alpha}}\omega(\rho) \le \omega(10\rho) \quad \text{for } \rho \in (0, r/10]. \tag{6.4}$$

Using the transformation $x \longrightarrow \rho^{-1} \cdot (x - x_0)$, we can consider only the case $x_0 = 0, \rho = 1$. In addition, replacing, if necessary, $V(x)$ by one of the functions $\lambda x_n \pm V(x)$, $\lambda = const$, we assume that

$$0 \le x_n^{-1} \cdot V(x) \le \omega = \omega(10) \quad \text{in } B^+_{10}, \tag{6.5}$$

and moreover,

$$|\{x \in B_2(x^*) : x_n^{-1} \cdot V(x) \ge \omega/2\}| \ge |B_2|/2, \tag{6.6}$$

where $x^* = (0, \cdots, 0, 4)$. Since $V \ge 0$ and $a_{ij}D_{ij}V = 0$ in $B_4(x^*) \subset B^+_{10}$, and by virtue of (6.6),

$$|\{x \in B_2(x^*) : V(x) \ge \omega\}| \ge |B_2|/2,$$

from Lemma 4.1 it follows that $V(x) \ge \beta\omega$ on $B_2(x^*)$, where $\beta = \beta(n,\nu) > 0$.

We fix $y = (y_1, \cdots, y_n) \in B^+_1$ and set

$$y^* = (y_1, \cdots, y_{n-1}, 4), \quad w(x) = (|x - y^*|^{-\gamma} - 4^{-\gamma}) \cdot \beta\omega,$$

where the constant $\gamma = \gamma(n,\nu) > 0$ is so large that $a_{ij}D_{ij}w(x) \ge 0$ for all $x \ne y^*$. Moreover, we have $V(x) \ge 0 = w(x)$ on $\partial B_4(y^*) \subset \overline{B^+_{10}}$, and

$V(x) \geq \beta\omega \geq w(x)$ on $\partial B_1(y^*) \subset B_2(x^*)$. Consequently, by the classical maximum principle, $V(x) \geq w(x)$ on $B_4(y^*) \setminus B_1(y^*)$. In particular,

$$V(y) \geq w(y) = ((4-y_n)^{-\gamma} - 4^{-\gamma})\beta\omega \geq \beta_1\omega \cdot y_n,$$

where $\beta_1 = \beta_1(n,\nu) > 0$. Since $y \in B_1$ can be selected in an arbitrary manner, we get $V(x)/x_n \geq \beta_1\omega$ for all $x \in B_1^+$. This estimate together with (6.5) yield $\omega(1) \leq (1-\beta_1)\cdot\omega(10)$. Taking $\overline{\alpha} = \overline{\alpha}(n,\nu) = -\log_{10}(1-\beta_1) > 0$, we obtain the desired inequality (6.3). □

**Remark 6.1.** By standard barrier technique, one can show that if (6.1), (6.2) are valid for $B_{2r}^+$ in place of $B_r^+$, then

$$\sup_{B_r^+} |V(x)/x_n| \leq N(n,\nu)\cdot r^{-1} \sup_{B_{2r}^+} |V|.$$

Therefore, in this case we have

$$\rho^{-\overline{\alpha}}\omega(\rho) \leq N r^{-1-\overline{\alpha}} \sup_{B_{2r}^+} |V| \quad \text{for all} \quad \rho \in (0,r].$$

We observe that since $V(x_0) = 0$, the estimate (6.3) yields the existence of the derivative

$$D_nV(x_0) = \lim_{\rho\to 0+} \rho^{-1}\cdot V(x_0+\rho e_n).$$

Lemma 6.1 can be applied to each point $y_0 \in \Gamma$ in place of $x_0$, hence there exists $D_nV$ on $\Gamma$ and moreover, the following assertions hold.

**Corollary 6.1.** *Under the assumptions of Lemma 6.1, we have*

$$[D_nV]_{\overline{\alpha};B^0_{r/2}} \leq N r^{-\overline{\alpha}}\omega(r), \tag{6.7}$$

*where* $B^0_{r/2} = B_{r/2}(x_0) \cap R^n_0$. *In addition, if* $D_nV \in C(\overline{B_r^+})$, *then*

$$[D_nV]_{\overline{\alpha};B^0_{r/2}} \leq N r^{-\overline{\alpha}} \operatorname*{osc}_{B_r^+} D_nV. \tag{6.8}$$

*Proof.* Applying (6.3) to different half-balls $B_\rho^+(y_0)$), we get (6.7). Further, if $D_nV \in C(\overline{B_r^+})$, then

$$V(x)/x_n = x_n^{-1}\int_0^{x_n} D_nV(x_1,\cdots,x_{n-1},t)\,dt,$$

Therefore,

$$\omega(r) = \operatorname*{osc}_{B_r^+} V(x)/x_n \leq \operatorname*{osc}_{B_r^+} D_nV, \tag{6.9}$$

that yields (6.8). □

**Corollary 6.2.** *Let the assumptions of Lemma 6.1 be satisfied and suppose that* $D_nV \in C(\overline{B_r^+})$. *Then*

$$\rho^{-1-\overline{\alpha}}E_1[V;B_\rho^+] \leq N r^{-\overline{\alpha}} \operatorname*{osc}_{B_r^+} D_nV \quad \text{for all} \quad \rho \in (0,r].$$

*Proof.* Since $\rho/x_n > 1$ in $B^+_\rho$, from Lemma 6.1 applied to the functions $V(x) - \lambda x_n$, $\lambda \in R^1$, we obtain:

$$\rho^{-1-\overline{\alpha}} E_1[V; B^+_\rho] \leq \rho^{-1-\overline{\alpha}} \inf_{\lambda \in R^1} \sup_{B^+_\rho} |V(x) - \lambda x_n|$$

$$\leq \rho^{-\overline{\alpha}} \inf_{\lambda \in R^1} \sup_{B^+_\rho} |V(x)/x_n - \lambda| = \frac{1}{2} \rho^{-\overline{\alpha}} \omega(\rho) \leq N r^{-\overline{\alpha}} \omega(r).$$

By virtue of (6.9), the desired estimate holds. □

## 7. Boundary $C^{2,\alpha}-$ estimates: the Dirichlet problem

In order to obtain $C^{2,\alpha}$-estimates of solutions near the boundary, with certain boundary conditions, we need appropriate extensions of Theorem 4.1. The following result of N.V. Krylov [12] can be treated as such an extension in the case of the Dirichlet boundary condition.

**Theorem 7.1.** *Let* $\nu \in (0, 1]$, $x_0 \in R^n_0$, $r > 0$, $B^+_r = B^+_r(x_0)$, $\varphi \in C(\overline{B^+_r})$, $\varphi = 0$ *on* $\Gamma = R^n_0 \cap B_r(x_0)$ *, and the function* $F_0(u_{ij}) \in \mathcal{F}(\nu)$*. Then the problem*

$$F_0[v] = F_0(D_{ij}v) = 0 \text{ in } B^+_r, \quad v = \varphi \text{ on } \partial B^+_r \tag{7.1}$$

*has a unique solution* $v \in C^{2,\overline{\alpha};0}(B^+_r \cup \Gamma) \cap C(\overline{B^+_r})$ *, and*

$$\|v\|^{(0)}_{2,\overline{\alpha};B^+_r \cup \Gamma} \leq N \cdot \sup_{\partial B^+_r} |\varphi|, \tag{7.2}$$

*where the constants* $\overline{\alpha} \in (0, 1]$, $N > 0$ *depend only on* $n, \nu$.

Our proof of this theorem is different from [12] and is based on Lemma 6.1. We precede it with the following auxiliary statement.

**Lemma 7.1.** *In addition to the assumption of Theorem 7.1, let the functions* $F_0 = F_0(u_{ij})$ *and* $v = v(x)$ *in (7.1) be smooth. Then for any* $\varepsilon > 0$ *and* $0 < \rho \leq r$*, we have*

$$\rho^{-\overline{\alpha}} \omega(x_0, \rho) \leq \varepsilon [v]_{2,\overline{\alpha};B^+_r} + N(\varepsilon, n, \nu) r^{-\overline{\alpha}} \omega(x_0, r), \tag{7.3}$$

*where*

$$\omega(x, \rho) = \max_{i,j} \operatorname*{osc}_{B^+_\rho(x)} D_{ij}v, \tag{7.4}$$

$B^+_r = B^+_r(x_0)$*, and* $\overline{\alpha} = \overline{\alpha}(n, \nu) \in (0, 1]$ *is the constant in Lemma 6.1.*

*Proof.* For $k = 1, 2, \cdots, n-1$, set $V^k = D_k v$. Since $v = \varphi = 0$ on $\Gamma \subset R^n_0$, we have also $V^k = 0$ on $\Gamma$. Further, differentiating the equality $F_0[v] = 0$ with respect to $x_k$, we obtain: $a_{ij} D_{ij} V^k = 0$ in $B^+_r$, where $a_{ij} = \partial F_0 / \partial u_{ij}$.

Applying to $V^k$ first Lemma 2.3 with $k=1$, and then Corollary 6.2, we find that

$$
\begin{aligned}
\rho^{-\overline{\alpha}} \operatorname*{osc}_{B_\rho^+} D_{ik}v &= \rho^{-\overline{\alpha}} \operatorname*{osc}_{B_\rho^+} D_i V^k \\
&\le \varepsilon [V^k]_{1,\overline{\alpha};B_\rho^+} + N(\varepsilon)\rho^{-1-\overline{\alpha}} E_1[V^k; B_\rho^+] \\
&\le \varepsilon [v]_{2,\overline{\alpha};B_r^+} + N(\varepsilon) r^{-\overline{\alpha}} \omega(x_0, r)
\end{aligned} \tag{7.5}
$$

for all $\varepsilon > 0$, $0 < \rho \le r$, and $i = 1, 2, \cdots, n$.

Since $k \le n-1$, on the left hand side of (7.5) there can appear any second derivative of $v$ except $D_{nn}v$. Further, by virtue of (3.11), for each $x, y \in B_\rho^+$ we have

$$
\overline{a}_{ij} \cdot (D_{ij}v(x) - D_{ij}v(y)) = 0
$$

with $\overline{a}_{ij}$ satisfying (3.2). Since $\overline{a}_{nn} \ge \nu > 0$, we get

$$
\operatorname*{osc}_{B_\rho^+} D_{nn}v \le N(n,\nu) \sum_{i+k \le 2n-1} \operatorname*{osc}_{B_\rho^+} D_{ik}v.
$$

From the last relation and (7.5), after redefining $\varepsilon$, the desired estimate follows. □

*Proof of Theorem 7.1.* Let $\alpha = \alpha(n,\nu) \in (0,1]$ be the smaller of the constants $\overline{\alpha}$ in Theorem 4.1 and Lemma 6.1. As in the proof of Theorem 4.1, we will assume without loss of generality that $F_0(u_{ij})$ is smooth and the problem (7.1) has a solution $v \in C^{2,\alpha;0}(B_r^+ \cup \Gamma) \cap C(\overline{B_r^+})$. The last assumption can be substantiated by the method of continuation with respect to the parameter $\nu$, which is outlined in Remark 5.1. Notice that from the smoothness of $F_0$ it follows $v \in C^3(B_r^+ \cup \Gamma)$ (see [9], Sec. 17.8).

We first prove that $\omega(x,\rho)$ in (7.4) satisfies

$$
d^{2+\alpha}(x)\rho^{-\alpha}\omega(x,\rho) \le \varepsilon V_{2,\alpha} + N(\varepsilon)V_2 \tag{7.6}
$$

for all $x \in B_r^+ \cup \Gamma$, $\rho \in (0, d(x)]$, and $\varepsilon > 0$, where in accordance with (2.9),

$$
d(x) = \frac{1}{2}\text{dist}(x, \partial\Omega \setminus \Gamma), \quad V_{2,\alpha} = [v]^{(0)}_{2,\alpha;B_r^+\cup\Gamma}, \quad V_2 = [v]^{(0)}_{2,0;B_r^+\cup\Gamma}.
$$

We will divide the proof of (7.6) into several cases.

(a) $x \in \Gamma$, $0 < \rho \le d(x)$. In this case (7.6) follows immediately from Lemma 7.1 with $x_0 = x, r = d(x)$.

(b) $d(x)/4 \le \rho \le d(x)$. Since $d^\alpha(x)\rho^{-\alpha} < 4^\alpha < 4$, the left hand side of (7.6) does not exceed $4d^2(x)\omega(x, d(x)) \le 8V_2$, so this estimate is true even with $\varepsilon = 0$.

(c) $x_n \le \rho \le d(x)/4$, where $x = (x', x_n)$, $x' \in \Gamma \subset R^n_0$. In this case we have $B_\rho^+(x) \subset B_{2\rho}^+(x')$. Moreover, (7.6) is valid for $x = x'$, hence $2\rho < d(x)/2 < d(x')$, and in view of (a) we get

$$
d^{2+\alpha}(x)\rho^{-\alpha}\omega(x,\rho) \le 2^{2+2\alpha} d^{2+\alpha}(x')(2\rho)^{-\alpha}\omega(x',\rho) \le 2^{2+2\alpha} \cdot [\varepsilon V_{2,\alpha} + N(\varepsilon)V_2],
$$

which, after redefining $\varepsilon$, also leads to (7.6).

(d) $0 < \rho < \rho_0 = \min(d(x)/4, x_n)$. First we apply the estimate (4.15); this gives us $\rho^\alpha \omega(x,\rho) \le N\rho_0^{-\alpha}\omega(x,\rho_0)$, and then (7.6) follows from (b) or (c), depending whether $\rho_0 = d(x)/4$ or $\rho_0 = x_n < d(x)/4$.

We have proved (7.6). By virtue of (2.19), we get

$$N_1^{-1} V_{2,\alpha} \le \hat{M}_{2,\alpha}^{(0)}[v; B_r^+ \cup \Gamma] \le \varepsilon V_{2,\alpha} + N(\varepsilon) V_2.$$

Setting $\varepsilon = (2N_1)^{-1}$, we get $V_{2,\alpha} \le NV_2$. Finally, using the interpolation inequalities (2.12), and then Corollary 3.3, we obtain

$$\|v\|_{2,\alpha;B_r^+\cup\Gamma}^{(0)} \le N \cdot \sup_{B_r^+} |v| = N \cdot \sup_{\partial B_r^+} |\varphi|,$$

so the desired inequality (7.2) is true with $\overline{\alpha} = \alpha = \alpha(n,\nu) \in (0,1]$. □

**Theorem 7.2.** *Let $F(x,u,u_i,u_{ij})$ satisfy Assumptions 3.1 with $\Omega = B_{r_0}^+(x_0)$ and some constants $K, K_1 \ge 0, \nu \in (0,1], \alpha \in (0,\overline{\alpha})$, where $x_0 \in R_0^n$, $r_0 \in (0,1]$, and $\overline{\alpha} = \overline{\alpha}(n,\nu)$ is the constant in Theorem 7.1. Let $u_0 \in C^{2,\alpha}(\Gamma)$, where $\Gamma = R_0^n \cap B_{r_0}(x_0)$ is identified with a ball in $R_0^n = R^{n-1}$. Then for any function $u \in C^{2,\alpha;0}(B^+ \cup \Gamma)$, satisfying the equalities*

$$F[u] = 0 \text{ in } B^+, \quad u = u_0 \text{ on } \Gamma, \tag{7.7}$$

*we have*

$$\|u\|_{2,\alpha;B^+\cup\Gamma}^{(0)} \le N(n,\nu,K,\alpha) \cdot \left[\sup_{B^+} |u| + r_0^{2+\alpha}(K_1 + |u_0|_{2,\alpha;\Gamma})\right]. \tag{7.8}$$

*Proof.* Setting

$$\hat{u} = u - u_0, \quad \hat{F}(x,u,u_i,u_{ij}) = F(x, u + u_0(x), u_i + D_i u_0(x), u_{ij} + D_{ij} u_0(x)),$$

one can see that the equalities (7.7) are equivalent to

$$\hat{F}[\hat{u}] = 0 \text{ in } B^+, \quad \hat{u} = 0 \text{ on } \Gamma.$$

Moreover, $\hat{F}$ satisfies Assumptions 3.1 with a new constant $\hat{K}_1 = N \cdot (K_1 + |u_0|_{2,\alpha;\Gamma})$ in place of $K_1$. Thus the proof of (7.8) is reduced to the case $u_0 = 0$.

As in the proof of Theorem 5.1, we introduce the notations

$$U_{2,\alpha} = [u]_{2,\alpha;B^+\cup\Gamma}^{(0)}, \quad U_k = [u]_{k,0;B^+\cup\Gamma}^{(0)},$$

and then we fix

$$y = (y', y_n) \in B^+ \cup \Gamma, \ d = d(y) = \frac{1}{2}\operatorname{dist}(y, \partial B^+ \setminus \Gamma), \ \rho \in (0,d], \ \varepsilon \in (0,1/2].$$

Then in the cases (a) $\rho/\varepsilon \le \min\{y_n, d/8\}$ and (b) $\rho/\varepsilon > d/8$, quite analogously to (5.5), we obtain the estimate

$$d^{2+\alpha}\rho^{-2-\alpha} E_2[u; B_\rho^+(y)] \le N\varepsilon^{\overline{\alpha}-\alpha} U_{2,\alpha} + N\varepsilon^{-2-\alpha}(U_2 + U_1 + U_0 + r_0^{2+\alpha} K_1). \tag{7.9}$$

In the remained case (c) $y_n < \rho/\varepsilon \le d/8$, we take $d' = \frac{1}{2}\operatorname{dist}\{y', \partial B^+ \setminus \Gamma\}$, $r = 4\rho/\varepsilon$, $\varphi = u - T_{y',2}u$, $\overline{u} = u(y')$, $\overline{u}_i = D_i u(y')$, $\overline{u}_{ij} = D_{ij}u(y')$, and $F_0(u_{ij}) = F(y', \overline{u}, \overline{u}_i, \overline{u}_{ij} + u_{ij}) \in \mathcal{F}(\nu)$. It is easy to see that

$$B_\rho^+(y) \subset B_{r/2}^+(y'), \quad r \le d/2 \le d'.$$

Moreover, since $u = 0$ on $\Gamma$ and $y \in \Gamma$, also $\varphi = 0$ on $\Gamma$. Define $v$ as the solution of the problem (7.1) in $B_r^+ = B_r^+(y')$. By analogy to (5.2), relying on Theorem 7.1 in place of Theorem 4.1, we have

$$d^{2+\alpha}\rho^{-2-\alpha}E_2[v; B_\rho^+(y)] \le N\varepsilon^{\overline{\alpha}-\alpha}U_{2,\alpha}.$$

The other points of the proof of Theorem 5.1, yielding the estimate (7.9) in the case (c) and the desired estimate (7.8), are valid with minimal modifications. □

**Remark 7.1.** Theorem 7.2 together with Theorem 3.1 yield Theorem 3.2. Indeed, upon dividing $\partial\Omega$ into a finite number of small subsets and "flattening of the boundary", Theorem 7.2 gives $C^{2,\alpha}$- estimates near $\partial\Omega$ for solutions of the problem (3.4). These estimates and interior $C^{2,\alpha}$-estimate (3.6) constitute the estimate (3.7) in Theorem 3.2.

## 8. Boundary $C^{2,\alpha}-$ estimates: the oblique derivative problem

The formulation of the following Theorems 8.1 and 8.2 are similar to ones of Theorems 7.1 and 7.2, only instead of the Dirichlet condition on $\Gamma$ we now have $D_n u = 0$ on $\Gamma$.

**Theorem 8.1.** *Let $\nu \in (0,1]$, $x_0 \in R_0^n$, $r > 0$, $B_r^+ = B_r^+(x_0)$, $\varphi \in C(\overline{B_r^+})$ be given, and the function $F_0(u_{ij}) \in \mathcal{F}(\nu)$. Then the equation*

$$F_0[v] = F_0(D_{ij}v) = 0 \text{ in } B_r^+ \tag{8.1}$$

*with the boundary conditions*

$$D_n v = 0 \text{ on } \Gamma = R_0^n \cap B_r(x_0), \quad v = \varphi \text{ on } \partial B_r^+ \setminus \Gamma, \tag{8.2}$$

*has a unique solution $v \in C^{2,\alpha;0}(B_r^+ \cup \Gamma) \cap C(\overline{B_r^+})$, and*

$$\|v\|^{(0)}_{2,\alpha;B_r^+\cup\Gamma} \le N \cdot \sup_{\partial B_r^+}|\varphi|, \tag{8.3}$$

*where the constants $\alpha \in (0,1]$, $N > 0$ depend only on $n, \nu$.*

*Proof.* We choose $\alpha \in (0, \overline{\alpha})$ (for example, $\alpha = \overline{\alpha}/2$), where $\overline{\alpha} = \overline{\alpha}(n,\nu) \in (0,1]$ be the smaller of the constants $\overline{\alpha}$ in Theorem 4.1 and Lemma 6.1. Under such choice of $\alpha$, we will prove (8.3).

*Step 1.* All the reasonings concerning the existence of the solution of the problem (8.1), (8.2) are quite similar to ones in the previous section related to the problem (7.1). Therefore, we will assume the existence of solution $v$, and moreover, we will consider smooth $F_0(u_{ij})$, so that $v \in C^3(B_r^+ \cup \Gamma)$. By Corollary 3.1, using the equalities (8.2), we get

$$V_0 = \sup_{B_r^+} |v| = \sup_{\partial B_r^+ \setminus \Gamma} |v| \le \sup_{\partial B_r^+} |\varphi|. \tag{8.4}$$

Following the lines of the proof of Theorem 7.1, we notice that it suffices to prove the estimate

$$d^{2+\alpha}(y)\rho^{-\alpha}\omega(y,\rho) \le N \cdot (V_2 + V_0) \tag{8.5}$$

for all $y \in B_r^+ \cup \Gamma$, $0 < \rho \le d(y) = \frac{1}{2}\mathrm{dist}(y, \partial\Omega \setminus \Gamma)$. This estimate is similar to (7.6), and the cases (b)–(d)) of its proof remain valid, so we will consider only the case (a) $y \in \Gamma$.

*Step 2.* Let us fix $y \in \Gamma$ and $d = d(y)$. Differentiating the equality (8.1) with respect to $x_n$, we obtain $a_{ij} D_{ij} D_n v = 0$ in $B_r^+$, where $a_{ij} = \partial F_0 / \partial u_{ij}$. Since $D_n v = 0$ on $\Gamma$, we can apply Corollary 6.1 to the function $V = D_n v$ in $B_d^+(y) \subset B_r^+$, that gives us the estimate

$$[D_{nn} v]_{\alpha; B^0_{d/2}(y)} \le N d^{-\alpha} \operatorname*{osc}_{B_d^+(y)} D_{nn} v \le N d^{-2-\alpha} V_2. \tag{8.6}$$

Besides this, we have $D_{in} v = 0$ on $\Gamma$ for $i = 1, \cdots, n-1$. Therefore, setting $\Omega = B^0_{d/2}(y) \subset R^n_0 = R^{n-1}$, we see that the function $v_0(x') = v_0(x_1, \cdots, x_{n-1}) = v(x', 0)$ satisfies the equality

$$F_0(D_{ij} v_0(x'), 0, \cdots, 0, D_{nn} v(x', 0)) = 0 \quad \text{in } \Omega.$$

By virtue of (8.6), the corresponding function

$$F(x', u_{ij}) = F_0(u_{ij}, 0, \cdots, 0, D_{nn} v(x', 0)) = 0 \quad \text{on } S^{n-1}$$

satisfies Assumptions 3.1 with $K = 0$, $K_1 = N d^{-2-\alpha} V_2$. This enables us to use Theorem 5.1, yielding the estimate

$$\|v_0\|^{(0)}_{2,\alpha; B^0_{d/2}(y)} \le N \cdot (V_0 + K_1 d^{2+\alpha}) \le N \cdot (V_0 + V_2). \tag{8.7}$$

*Step 3.* Now we can apply Theorem 7.2 with $B^+ = B^+_{d/2}(y)$, $\Gamma = B^0_{d/2}(y)$. This gives us

$$d^{2+\alpha}[v]_{2,\alpha; B^+_{d/4}(y)} \le 4^{2+\alpha}[v]^{(0)}_{2,\alpha; B^+ \cup \Gamma} \le N \cdot \left( V_0 + \|v_0\|^{(0)}_{2,\alpha;\Gamma} \right) \le N \cdot (V_0 + V_2).$$

The last estimate contains (8.5) for $0 < \rho \le d/4$. If $d/4 < \rho \le d = d(y)$, then

$$d^{2+\alpha}\rho^{-\alpha}\omega(y,\rho) \le 4^\alpha d^2 \omega(y,\rho) \le N V_2,$$

so (8.5) is true for all $\rho \in (0, d]$. □

**Theorem 8.2.** *Let $\nu \in (0,1]$, $x_0 \in R^n_0$, $r_0 > 0$, $B^+ = B^+_{r_0}(x_0)$, $\Gamma = R^n_0 \cap B_{r_0}(x_0)$, and let $F(x, u, u_i, u_{ij})$ satisfy Assumptions 3.1 with $\Omega = B^+$ and some* *constants* $K, K_1 \geq 0, \alpha \in (0,1)$*. Then for any function* $u \in C^{2,\alpha;0}(B^+ \cup \Gamma)$ *satisfying the equalities*

$$F[u] = 0 \ \text{ in } \ B^+, \quad D_n u = 0 \ \text{ on } \ \Gamma, \tag{8.8}$$

*we have*

$$\|u\|^{(0)}_{2,\alpha;B^+\cup\Gamma} \leq N \cdot \left( \sup_{B^+} |u| + r_0^{2+\alpha} K_1 \right), \tag{8.9}$$

*where* $N = N(n, \nu, K, \alpha)$*, provided* $0 < \alpha < \overline{\alpha}$ *for some constant* $\overline{\alpha} = \overline{\alpha}(n, \nu) \in (0,1)$.

*Proof.* We take as $\overline{\alpha} = \overline{\alpha}(n,\nu) \in (0,1)$ the constant $\alpha$ in Theorem 8.1. Then we can reproduce almost literary the proof of Theorem 7.2, only in the case (c) we define $v$ as the solution of the problem (8.1), (8.2), and accordingly, we rely on Theorem 8.1 instead of Theorem 7.1

Repeating the reasonings in Step 2 of the proof of Theorem 5.1, we now have

$$F_0[\varphi] = F_0[\varphi] - F_0[v] = L_0(\varphi - v) = a_{ij} D_{ij}(\varphi - v) \quad \text{in} \ \ B_r^+(y')$$

with $a_{ij}$ satisfying (3.2), and the function $v(x)$ together with

$$\varphi(x) = u(x) - T_{y',2}u(x), \quad w(x) = \frac{A r^\alpha}{2n\nu}(r^2 - |x - y'|^2)$$

satisfy the relations

$$|L_0(\varphi - v)| \leq r^\alpha A = r^\alpha \sup_{B_r^+(y')} |F_0[\varphi]| \leq -L_0 w \ \text{ in } \ B_r^+(y'),$$

$$D_n(\varphi - v)| = D_n w = 0 \ \text{ on } \ \Gamma \cap B_r(y'), \quad \varphi - v = w = 0 \ \text{ on } \ \partial B_r^+(y') \setminus \Gamma.$$

Therefore, we can apply the comparison principle yielding

$$\sup_{B_\rho^+(y')} |\varphi - v| \leq \sup_{B_r^+(y')} |\varphi - v| \leq \sup_{B_r^+(y')} |w| = \frac{A}{2n\nu} r^{2+\alpha}.$$

The remained part of the proof is almost the same as in the proof of Theorem 5.1 , so we obtain the estimate (8.9). □

We will use the estimate (8.9) in the proof of the $C^{2,\alpha}$-estimate (3.10) in the formulation of the Theorem 3.3. However, "flattening of the boundary" $\partial\Omega \in C^{1,\alpha}$ in the *general case* would deteriorate $C^{2,\alpha}$-smoothness of solutions. Therefore, we first consider a *special case* of the boundary conditions which is reduced to (8.8). For this purpose, we need some auxiliary results concerning the extension of functions from $\partial\Omega$ to $\Omega$. The following lemma is contained in [8], Lemma 2.3.

**Lemma 8.1.** *Let* $r > 0, B_r^+ = R_+^n \cap B_r(0), B_r^0 = R_0^n \cap B_r(0)$, *and* $\Phi_0 \in C^{1,\alpha}(B_r^0), 0 < \alpha < 1$. *Then there exists a function* $\Phi \in C^\infty(B_r^+) \cap C^{1,\alpha}(\overline{B_r^+})$ *such that* $\Phi = \Phi_0$ *on* $B_r^0$,

$$(8.10) \qquad |\Phi|_{1,\alpha;B_r^+} \le N(n,\alpha) \cdot |\Phi_0|_{1,\alpha;B_r^0},$$

*and for any* $k = 2, 3, \cdots$, *we have*

$$(8.11) \qquad \max_{|l|=k} \sup_{B_r^+} y_n^{k-1-\alpha} |D^l \Phi(y)| \le N(k,n,\alpha) \cdot [\Phi_0]_{1,\alpha;B_r^0}.$$

Let us fix an arbitrary point $x_0 \in \partial\Omega \in C^{1,\alpha}$. Using Definition 3.1 and the implicit theorem, we cam choose an orthonormal coordinate system centered at $x_0$ and $r_0 \in (0,1]$ such that $\Omega_{r_0}(x_0) = \Omega \cap B_{r_0}(x_0)$ is represented in the form (2.23) with $\psi_0 \in C^{1,\alpha}(B_{r_0}^0)$, where $B_{r_0}^0$ is the projection of $B_{r_0}(x_0)$ onto $R_0^n$.

Without loss of generality we now take $x_0 = 0$. Applying Lemma 8.1 with $\Phi_0 = \psi_0$, $0 < r \le r$, we obtain the existence of a smooth function $\psi \in C^\infty(B_r^+) \cap C^{1,\alpha}(\overline{B_r^+})$ such that $\psi = \psi_0$ on $B_r^0$, and (8.10), (8.11) are true for $\Phi = \psi, \Phi_0 = \psi_0$. Moreover, replacing $\psi(y)$ by $\psi(y) + Ny_n$ if necessary, we may assume that $D_n\psi \ge 1$ on $B_r^+$. Now we introduce the new coordinates $x = x(y) \in C^{1,\alpha}(\overline{B_r^+})$ by the mapping

$$x' = y', \;\; x_n = \psi(y), \;\; y \in B_r^+.$$

We have $\det \partial x/\partial y = D_n\psi \ge 1$, therefore, the inverse mapping $y = y(x)$, where

$$y' = x', \;\; y_n = \eta(x), \;\; x \in \Omega_r = x(B_r^+),$$

has the same smoothness as $x = x(y)$. It is easy to see that

$$N^{-1} y_n \le d(x) = \frac{1}{2}\text{dist}(x, \partial\Omega) \le N y_n = N\eta(x), \;\; x \in \Omega_r,$$

with $N = N(\Omega)$. Therefore, for $y_n = \eta(x)$ we have

$$(8.12) \qquad \max_{|l|=k} |D^l \eta(x)| \le N(k,\alpha,\Omega) \cdot y_n^{1+\alpha-k}, \quad x \in \Omega_r, \;\; k = 2, 3, \cdots.$$

Using all these properties, one can obtain the following lemma as a consequence of Lemma 8.1.

**Lemma 8.2.** *Let* $\Omega$ *be a bounded domain in* $R^n$ *with* $\partial\Omega \in C^{1,\alpha}$, $0 < \alpha < 1$. *Under the previous assumptions, let* $r \in (0, r]$ *be chosen small enough, so that* $\Omega_r = x(B_r^+) \subset \Omega$, *and let a function* $\phi_0 \in C^{1,\alpha}(\overline{\Omega_r})$ *be given. Then there exists a function* $\phi \in C^\infty(\Omega_r) \cap C^{1,\alpha}(\overline{\Omega_r})$ *such that* $\phi = \phi_0$ *on* $\gamma_r = \partial\Omega \cap \partial\Omega_r$,

$$(8.13) \qquad |\phi|_{1,\alpha;\Omega_r} \le N(\alpha,\Omega) \cdot |\phi_0|_{1,\alpha;\Omega_r},$$

*and for any* $k = 2, 3, \cdots$, *we have*

$$(8.14) \qquad \max_{|l|=k} \sup_{\Omega_r} d^{k-1-\alpha}(x) |D^l \phi(x)| \le N(k,\alpha,\Omega) \cdot [\phi_0]_{1,\alpha;\Omega_r}.$$

Notice that in this construction, we can take $r > 0$ independent on $x_0 \in \partial\Omega$. Using then the standard partition of unity (see [9], Sec. 6.9), we arrive at the following statement.

**Corollary 8.1.** *If* $\phi_0 \in C^{1,\alpha}(\overline{\Omega})$, *then there exists a function* $\phi \in C^\infty(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ *such that* $\phi = \phi_0$ *on* $\partial\Omega$,

$$|\phi|_{1,\alpha;\Omega} \le N(\alpha,\Omega)|\phi_0|_{1,\alpha;\Omega}, \tag{8.15}$$

*and for any* $k = 2, 3, \cdots$, *we have*

$$[\phi]^{(-1-\alpha)}_{k,o;\Omega} \le N(k,\alpha,\Omega)[\phi_0]_{1,\alpha;\Omega}. \tag{8.16}$$

The following two lemmas serve as the intermediate steps in the proof of the estimate (3.10). As before, we fix $x_0 \in \partial\Omega \in C^{1,\alpha}$, $0 < \alpha < 1$, and a suitable $C^{1,\alpha}$-mapping $x = x(y)$, so that some portion of $\Omega$ near $x_0$ is represented in the form $\Omega_r = x(B_r^+)$, $0 < r \le r_0$, and $\gamma_r = \partial\Omega \cap \partial\Omega_r = x(\Gamma)$, $\Gamma = B_r^0$.

**Lemma 8.3.** *Let* $F(x, u, u_i, u_{ij})$ *satisfy Assumptions 3.1 with some constants* $K, K_1 \ge 0$, $0 < \alpha < 1$. *Then for any function* $u \in C^{2,\alpha;0}(\Omega_r \cup \gamma_r)$ *satisfying the equalities*

$$F_0[u] = 0 \text{ in } \Omega_r, \quad D_n u = 0 \text{ on } \gamma_r = \partial\Omega \cap \partial\Omega_r, \tag{8.17}$$

*we have*

$$\|u\|^{(0)}_{2,\alpha;\Omega_r\cup\gamma_r} \le N(n,\nu,K,\alpha,\Omega)\cdot\left(\sup_{\Omega_r}|u| + r^{2+\alpha}K_1^*\right), \tag{8.18}$$

*where* $K_1^* = K_1 + |u|_{2,0;\Omega_r}$, *provided* $0 < \alpha < \overline{\alpha}$ *for some constant* $\overline{\alpha} = \overline{\alpha}(n,\nu,\Omega) \in (0,1)$.

*Proof.* Under the $C^{1,\alpha}$-diffeomorphism

$$x \in \Omega_r \longleftrightarrow y \in B_r^+, \text{ where } x' = y',\ x_n = \psi(y),\ y_n = \eta(x), \tag{8.19}$$

let us define $\hat{u}(y) = u(x)$. We have

$$D_i u(x) = D_k\hat{u}(y)\cdot D_i y_k(x),$$

$$D_{ij}u(x) = D_{km}\hat{u}(y)\cdot D_i y_k(x)\cdot D_j y_m(x) + D_n\hat{u}(y)\cdot D_{ij}\eta(x). \tag{8.20}$$

From (8.17) it folows

$$\hat{F}[\hat{u}(y)] = \hat{F}(y,\hat{u},D_k\hat{u},D_{km}\hat{u}) = 0 \text{ in } B_r^+, \quad D_n\hat{u} = 0 \text{ on } \Gamma = B_r^0,$$

where

$$\hat{F}(y,u,u_k,u_{km}) = F(x,u,f_{ik}u_k, f_{ik}f_{jm}u_{km} + g_{ij}),$$
$$f_{ik} = f_{ik}(y) = D_i y_k(x), \quad g_{ij} = g_{ij}(y) = D_n\hat{u}(y)\cdot D_{ij}\eta(x).$$

Since $D_n\hat{u}(y',0) = 0$, by the mean value theorem we have $|D_n\hat{u}(y)| \le y_n\,[\hat{u}]_{2,0}$. Together with (8.12), this gives us the estimates

$$|g_{ij}(y)| \le N y_n^{\alpha} \cdot [\hat{u}]_{2,0}, \quad |Dg_{ij}(y)| \le N y_n^{\alpha-1} \cdot [\hat{u}]_{2,0}.$$

Applying Lemma 2.4, we get

$$[g_{ij}]_{\alpha} \le N\,[g_{ij}]_{1,0}^{(-\alpha)} \le N\,[\hat{u}]_{2,0},$$

hence

$$|f_{ik}|_{\alpha} \le N, \quad |g_{ij}|_{\alpha} \le N\,[\hat{u}]_{2,0}.$$

Relying on this estimates, it is easy to show that Assumptions 3.1 on the function $F$ yield the similar assumptions on $\hat{F}$, with $\Omega = B_r^+$ and some constants

$$\hat{\nu} \ge \nu/N, \quad \hat{K} \le NK, \quad \hat{K}_1 \le N \cdot (K_1 + [\hat{u}]_{2,0})$$

in place of $\nu, K, K_1$. By Theorem 8.2 we can assert

$$\|\hat{u}\|_{2,\alpha;B^+\cup\Gamma}^{(0)} \le N \cdot \left( \sup_{B^+} |\hat{u}| + r_0^{2+\alpha}\hat{K}_1 \right).$$

Furthermore, (8.20) brings us to the estimates

$$[u]_{2,0}) \le N \cdot [\hat{u}]_{2,0}, \quad \|u\|_{2,\alpha;\Omega_r\cup\gamma_r}^{(0)} \le N \cdot \|\hat{u}\|_{2,\alpha;B^+\cup\Gamma}^{(0)}.$$

Since the mapping $x = x(y)$ has the same properties as $y = y(x)$, we also have $[\hat{u}]_{2,0} \le N\,[u]_{2,0}$. These inequalities provide us the estimate (8.18). □

**Lemma 8.4.** *In the formulation of Lemma 8.3, replace (8.17) with the equalities*

$$F_0[u] = 0 \text{ in } \Omega_r, \quad D_n u = g_0 \text{ on } \gamma_r, \tag{8.21}$$

*where* $g_0 \in C^{1,\alpha}(\overline{\Omega_r})$. *Then the estimate (8.18) remains valid with*

$$K_1^* = K_1 + |u|_{2,0;\Omega_r} + G_0, \quad \text{where } G_0 = |g_0|_{1,\alpha;\Omega_r}.$$

*Proof.* Combining Lemma 8.2 with the standard extension lemmas (see [9], Sec. 6.9), we can construct a function $g_1$ defined in a wider domain

$$Q_r = \{x = (x', x_n) \in R^n : x' \in B_r^0, \psi_0(x') < x_n < h\} \supset \Omega_r, \text{ where } h = const,$$

so that $g_1 \in C^{\infty}(Q_r) \cap C^{1,\alpha}(\overline{Q_r})$,

$$g_1 = g_0 \text{ on } \gamma_r = \partial\Omega \cap \partial\Omega_r = \{x \in R^n : x' \in B_r^0, x_n = \psi_0(x')\} \subset \partial Q_r,$$

and for any $k = 2, 3, \cdots$, we have

$$\max_{|l|=k} \sup_{Q_r} d^{k-1-\alpha}(x)|D^l g_1(x)| \le N(k,\alpha,\Omega) G_0. \tag{8.22}$$

We also have

$$N^{-1}d(x) \le x_n - \psi_0(x') \le Nd(x), \quad x \in Q_r. \tag{8.23}$$

Now we define

$$u_0(x) = u_0(x', x_n) = -\int_{x_n}^{h} g_1(x', t)\, dt, \quad x \in Q_r. \tag{8.24}$$

We state that

$$|D^l u_0(x)| \le N d^{\alpha-1}(x) G_0 \quad \text{for all} \quad |l| = 3,\ x \in Q_r. \tag{8.25}$$

If $|l| = 3, l_n > 0$, then $D^l u_0(x) = D_{ij} D_n u_0(x) = D_{ij} g_1(x)$ for some $i, j$, and (8.25) gives us (8.22). If $|l| = 3, l_n = 0$, then (8.24), (8.22) yield

$$|D^l u_0(x)| = \left| \int_{x_n}^{h} D^l g_1(x', t)\, dt \right| \le N G_0 \int_{x_n}^{h} d^{\alpha-2}(x', t)\, dt,$$

and (8.25) follows from (8.23). Finally, applying Lemma 2.4 and using (8.25), we obtain:

$$[u_0]_{2,\alpha} = \max_{i,j} [D_{ij} u_0]_\alpha \le N \max_{i,j} [D_{ij} u_0]^{-\alpha}_{1,0} \le N G_0.$$

Moreover, $D_n u_0 = g_1 = g_0$ on $\gamma_r$. By setting $\hat{u} = u - u_0$, as in the proof of Theorem 7.2, this lemma is reduced to Lemma 8.3. □

**Theorem 8.3.** *Under the assumptions of Theorem 3.3, let $u \in C^{2,\alpha}(\overline{\Omega})$ be a solution of the problem (3.9). Then the estimate (3.10) holds.*

*Proof.* Let us fix $x_0 \in \partial\Omega$. In the previous construction, we can choose an orthonormal coordinate system with $b(x_0) = (b_1(x_0), \cdots, b_n(x_0))$ directed along the positive $x_n$- axis. By virtue of (3.8), we can impose the restriction $N^{-1} \le \det \partial x/\partial y \le N$, where $N = N(n, \nu_0)$, on the $C^{1,\alpha}$-diffeomorphism (8.19). Therefore, the constant $\overline{\alpha}$ in Lemma 8.3 depends only on $n, \nu, \nu_0$.

Dividing both sides of the condition $b_i D_i u + b_0 u = g$ by $b_n(x_0) > 0$, we can reduce it to the case $b_i(x_0) = \delta_{in}$. Next, we rewrite it in the form

$$D_n u = g_0 = g - (b_i - \delta_{in}) D_i u - b_0 u \quad \text{on} \quad \gamma_r = \partial\Omega \cap \partial\Omega_r.$$

Since $|b_i(x) - \delta_{in}| = |b_i(x) - b_i(x_0)| \le N r^\alpha$ in $\Omega_r$, by virtue of (2.6) we get

$$|g_0|_{1,\alpha;\Omega_r} \le |\varphi|_{1,\alpha} + N r^\alpha U_{2,\alpha} + N |u|_{2,0}.$$

where $U_{2,\alpha} = [u]_{2,\alpha;\Omega}$. Then Lemma 8.4 gives us

$$\|u\|^{(0)}_{2,\alpha;\Omega_r \cup \gamma_r} \le N r^{2+2\alpha} U_{2,\alpha} + N(r) \cdot (K_1 + |\varphi|_{1,\alpha} + |u|_{2,0}). \tag{8.26}$$

Further, let us fix $\delta = 1/N > 0$ such that

$$\operatorname{dist}(\omega_r,\ \partial\Omega_r \setminus \gamma_r) \ge \delta r, \quad \text{where} \quad \omega_r = \Omega \cap B_{\delta r}(x_0).$$

The estimate (8.26) yields

$$[u]_{2,\alpha;\omega_r} \le N_0 r^\alpha U_{2,\alpha} + N(r)\cdot(K_1 + |\varphi|_{1,\alpha} + |u|_{2,0}), \tag{8.27}$$

where $N_0$ does not depend on $r > 0$.

Using the last estimate with arbitrary $x_0 \in \partial\Omega$, we will show that

$$U_{2,\alpha} = [u]_{2,\alpha;\Omega} \le N\cdot(K_1 + |\varphi|_{1,\alpha} + |u|_{2,0}). \tag{8.28}$$

By definition of $[u]_{2,\alpha}$, we can choose $x, y \in \Omega$, and $i, j$, such that

$$U_{2,\alpha} \le 2\,|D_{ij}u(x) - D_{ij}u(y)|/|x-y|^\alpha. \tag{8.29}$$

We consider separately three cases.

(a) $|x-y| < \delta r/3$, $d(x) = dist(x, \partial\Omega) < \delta r/3$. In this case for some $x_0 \in \partial\Omega$ we have $|x - x_0| = 2d(x) < 2\delta r/3$, hence $x, y \in \Omega \cap B_{\delta r}(x_0) = \omega_r$. Now let us fix $r > 0$ such that $N_0 r^\alpha < 1/4$ in (8.27). Since the right hand side in (8.29) does not exceed $[u]_{2,\alpha;\omega_r}$, from (8.27) we obtain (8.28).

(b) $|x-y| < \delta r/3$, $d(x) \ge \delta r/3$. We have $y \in B(x) = B_{d(x)}(x)$, hence (8.28) follows from the interior estimate (3.6).

(c) $|x-y| \ge \delta r/3$. Directly from (8.29) it follows $U_{2,\alpha} \le N\,|u|_{2,0}$.

We have proved (8.28). Finally, using the interpolation inequalities which are true even for Lipschitz domains (see [20], Ch.5, Sec.33), we can replace $|u|_{2,0}$ by $U_0 = \sup_\Omega |u|$ in (8.28), so that (8.28) turns into (3.10). □

**Remark 8.1.** Under the assumptions of Theorem 3.3, $U_0 = \sup_\Omega |u|$ is easy estimated by the comparison principle (see [9] the proof of Theorem 6.31). Hence we have a priori estimates of solutions to the oblique derivative problem (3.9) in $C^{2,\alpha}(\overline{\Omega})$, depending only on the prescribed constants. On the grounds of these $C^{2,\alpha}$-estimates, the solvability of the problem (3.9) can be stated by means of the satndard continuity method (see [9], Sec. 17.9).

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