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AN ESTIMATE OF THE PROBABILITY THAT A DIFFUSION PROCESS HITS A SET OF POSITIVE MEASURE

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1. Estimates of the hitting probabilities of one set or another play an important role in many questions in the theory of random processes. In the present article, we estimate for processes of diffusion type a similar probability from below in terms of the measure of a set. With the help of this estimate, we are able to prove that diffusion processes corresponding to parabolic operators with measurable coefficients possess the property of standardness (see the definition in [1], §3.23). This estimate can also be used in the derivation, for nondivergence equations, of twenty-year-old results of De Giorgi and Nash about Hölder continuity of solutions to elliptic and parabolic equations in divergence form (see [2], [3]). We remark that, for nondivergence elliptic and parabolic equations, under some restrictions on the dispersion of the eigenvalues of the matrix of coefficients of the higher derivatives, the theorems of De Giorgi and Nash were proved in [4]. With the help of our probability estimates, such theorems may be obtained without the restrictions on the magnitude of the dispersion of the eigenvalues.

2. Let $E_n = \{x = (x^1, \dots, x^n)\}$ be n-dimensional Euclidean space, $n, \mu, \nu, \theta, \alpha$ and ϵ be fixed numbers, with $\nu > \mu > 0$, $\theta \in (0, 1]$, $\alpha \in (0, 1)$, $\epsilon \in (0, 1)$. For $x \in E_n$ and $R > 0$, we set $|x| = \max|x^i|$ and $Q(\theta, R) = (0, \theta R^2) \times (|x| < R)$. Let $(\Omega, \mathcal{F}, P)$ be a probability space on which an n_1-dimensional Wiener process $(w_t, \mathcal{F}_t)$ is defined for some $n_1 \geqslant n$. For $t \geqslant 0$ and $\omega \in \Omega$ an $n \times n_1$ matrix σ_t and $b_t \in E_n$ are defined, where it is assumed that σ_t and b_t are progressively measurable with respect to the family $\{\mathcal{F}_t\}$ of σ-algebras and that there exists a number $R > 0$ such that $R|b_t| \leqslant \nu$ and $\nu|\lambda| \geqslant |\sigma_t^*\lambda| \geqslant \mu|\lambda|$ for all $\lambda \in E_n$, t, ω. We fix a suitable R. Finally, let $\beta \in [\theta, 1]$ and let $\Gamma \subset Q(\beta, R)$ be a closed set such that $|\Gamma| \geqslant \epsilon|Q(\beta, R)|$, where $|D|$ for $D \subset E_{n+1} = \{(t, x)\colon t \in E_1, x \in E_n\}$ denotes the Lebesgue measure of D. We take an $\mathcal{F}_0$-measurable variable $\xi \in E_n$ such that $P\{|\xi| > \alpha R\} = 1$, and we define a process x_t by means of the conditions $x_0 = \xi$, $dx_t = \sigma_t dw_t + b_t dt$. For $D \subset E_{n+1}$, we set $\tau(D) = \inf\{t > 0\colon (t, x_t) \in \partial D\}$ ($\inf \varnothing = \infty$). The main theorem of this paper is as follows.

THEOREM 1. *There exists a constant $\delta > 0$, depending only on $n, \mu, \nu, \theta, \alpha, \epsilon$ (but not on $\Gamma, R, \beta, \dots$), such that $P\{\tau(\Gamma) < \tau(Q(\beta, R))\} \geqslant \delta$.*

3. Let $Lu = u_t + a^{ij}u_{x^ix^j} + b^iu_{x^i} - cu$, where $a = (a^{ij})$, $b = (b^i)$, c is a Borel function from (t, x) to E_{n+1}, a is a symmetric matrix of dimension $n \times n$, $b \in E_n$ and $c \geqslant \mu$. Let $\nu|\lambda|^2 \geqslant (a\lambda, \lambda) \geqslant \mu|\lambda|^2$, $|b| \leqslant \nu$, $|c| \leqslant \nu$ for all $\lambda \in E_n$, t, x. Let $Y = (y_t, \zeta, N_t, P_y)$ be a continuous homogeneous Markov process in E_n, corresponding to L in the sense that, for

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arbitrary $u \in C_0(E_{n+1})$ and all $t > s$, $x \in E_n$,

$$u(s, x) = M_{(s, x)}\left[\int_0^{t-s} Lu(y_r)\,dr + u(y_{t-s})\right].$$

In addition, let N_t be the completion of $\sigma(y_s, s \leqslant t)$ with respect to all P_y. The existence of such a process follows, for example, from [5]. If the region $D \subset E_{n+1}$ and f is a Borel function, let τ be the first exit time of y_t from D:

$$R(D)f(y) = M_y \int_0^\tau f(y_t)\,dt, \quad \pi(D)f(y) = M_y f(y_\tau).$$

THEOREM 2. a) *Y is a strong Markov process and $N_t = N_{t+}$.*

b) *If $f \in \mathcal{L}_{n+1}(D)$ and $\varphi \in B(\partial D)$, then $R(D)f$ and $\pi(D)f$ are Hölder continuous on each compact set $K \subset D$, and also the Hölder exponent depends only on n, ν and μ, and a is a constant depending only on n, ν, μ, the distance from K to ∂D, and on the norms of f in $\mathcal{L}_{n+1}(D)$ and φ in $B(\partial D)$.*

4. THEOREM 3. *Let D be a region in E_{n+1} and $\mathcal{L}$ be the operator from §3. Then there exist bounded operators R: $\mathcal{L}_{n+1}(D) \longrightarrow B(D)$ and π: $B(\partial D) \longrightarrow B(D)$, taking nonnegative functions into nonnegative functions and such that* a) *for arbitrary $u \in W^{1,2}_{n+1}(D) \cap C(\bar{D})$ in D the representation $u = Rf + \pi\varphi$ holds, where $f = -Lu$ and φ is the value of u on ∂D;* b) *if $f_1, f_2 \in \mathcal{L}_{n+1}(D)$ and $Rf_1 = Rf_2$ (a.s.), then $f_1 = f_2$ (a.s.);* c) *assertion* b) *of Theorem 2 holds for R and π.*

An analogous theorem holds in the case when L is an elliptic operator (in the description of L in §3, u_t is absent and a, b and c do not depend on t). In order to formulate such a theorem, it suffices to replace, in Theorem 3, $n + 1$ by n and $W^{1,2}$ by W^2.

5. In the proofs of Theorems 2 and 3, Theorem 1 plays an important role. To clarify this, we outline the proof of the estimate of the Hölder exponent and constant of the solution in $W^{1,2}_{n+1}(D)$ to the equation $u_t + a^{ij}u_{x^ix^j} = 0$ (a.s. D), where a satisfies the conditions of §3. We set $Q(R) = Q(1, R)$. As is well known from the theory of differential equations (see [3], Chapter V, §10; [4], Chapter III, §8), to obtain the required estimate it suffices to show that $\operatorname{osc}\{u, Q(R)\} \leqslant \rho \cdot \operatorname{osc}\{u, Q(2R)\}$ (for $Q(2R) \subset D$) with a constant $\rho < 1$, depending only on n, ν, and μ. Furthermore, considering a fixed R, we may assume that $\max\{u, \bar{Q}(R)\} = 1$, $\min\{u, \bar{Q}(R)\} = -1$, and $2 \cdot |\Gamma| \geqslant |Q(2R)|$ for $\Gamma = \{u \leqslant 0\} \cap \bar{Q}(2R)$.

Let $(t_0, x_0) \in \bar{Q}(2R)$, $u(t_0, x_0) = 1$. On some probability space we find a solution to the equation $dx_t = [2a(t, x_t)]^{1/2}dw_t$, $t \geqslant t_0$, starting from x_0 for $t = t_0$, and let $\tau(\gamma)$ be the first time after t_0 that $\partial Q(2R)(\Gamma)$ is reached by the process (t, x_t). By Itô's formula, $1 = u(t_0, x_0) = Mu(\tau\Lambda\gamma, x_{\tau\Lambda\gamma}) \leqslant P\{\tau < \gamma\} \cdot \max\{u, Q(2R)\}$, where $\tau\Lambda\gamma = \min(\tau, \gamma)$. Since $|x_0| \leqslant R$, $0 \leqslant t_0 \leqslant R^2$ and $4 \cdot |\Gamma \cap \{t \geqslant t_0\}| \geqslant |Q(2R) \cap \{t \geqslant t_0\}|$, it follows from Theorem 1 for $\theta = 3/4$, $\alpha = 1/2$, and $\epsilon = 1/4$ that we have $P\{\gamma < \tau\} \geqslant \delta > 0$, where δ depends only on n, ν, and μ. Therefore $P\{\tau < \gamma\} \leqslant 1 - \delta$ and $1 \leqslant (1 - \delta)\max\{u, Q(2R)\}$. In addition, $\min\{u, Q(2R)\} \leqslant -1$ from the maximum principle. From this $2 = \operatorname{osc}\{u, Q(R)\} \leqslant (1 - \delta/2)\operatorname{osc}\{u, Q(2R)\}$, as required.

6. In the proof of Theorem 1, we make use of the following lemma from measure theory. Let $Q = Q(1, 1)$, let Γ be a measurable subset of Q, and let $r, q \in (0, 1)$. We denote by $\mathfrak{A}$ the system of all sets K of the form $(t, x) + Q(1, R)$ such that $K \subset Q$, $|K \cap \Gamma| \geqslant q|K|$. If $K \in \mathfrak{A}$ and $K = (t, x) + Q(1, R)$, then we set $K' = \{(t - 3R^2, x) + Q(7/9, 3R)\}$

$\cap Q$ and $K'' = \{(s, y): 0 < r(t - s) < 4R^2, |x - y| < 3R, |y| < 1\}$. Finally, let $D' = \bigcup K'$ and $D'' = \bigcup K''$, where the unions are taken over all $K \in \mathfrak{A}$.

LEMMA. *The sets D' and D'' are open, and if $|\Gamma| \leqslant q|Q|$, then $|\Gamma| \leqslant q|D'| \leqslant q(1 + r)|D''|$.*

7. We mention briefly the main features of the proof of Theorem 1. First of all, by a change of time and coordinates we achieve $\theta = R = 1$. For fixed values of n, μ, ν, α we vary ϵ within the interval $[0, 1)$. It is clear that if on the interval $[0, \tau(\partial Q)]$ the process leads in a short time to $Q \backslash \Gamma$, then over a longer time period it leads to Γ, and hence it reaches Γ. These considerations, together with Theorem II.2.4 of [6], demonstrate that we can find a $q \in (0, 1)$ such that, for $\epsilon \geqslant q^2$, the desired estimate holds with some $\delta^1 > 0$.

Let ϵ' be an upper bound of those ϵ for which the desired estimate holds only with $\delta = 0$. We remark that $\epsilon' \leqslant q$, and we have to show that $\epsilon' = 0$. Assume the contrary: $\epsilon' > 0$. Then we can find an $\epsilon'' < \epsilon'$ and $r \in (0, 1)$ such that $\epsilon' q(1 + r) < \epsilon''$. We define ϵ^0 from the equation $\epsilon'' = (2\epsilon^0 - \epsilon')q(1 + r)$. Since $\epsilon^0 > \epsilon'$, for $|\Gamma| \geqslant \epsilon^0 |Q|$ the desired estimate holds with some $\delta^0 > 0$. We now take Γ so that $|\Gamma| \geqslant \epsilon''|Q|$. From Γ and q we construct the set D'' of the lemma and consider two cases: 1) $|D'' \backslash D| < 2^{-1}(\epsilon^0 - \epsilon')|Q|$, and 2) $|D'' \backslash D| \geqslant 2^{-1}(\epsilon^0 - \epsilon')|Q|$.

In the first case, it follows from the lemma that $|D'' \cap D| > [\epsilon^0 + 2^{-1}(\epsilon^0 - \epsilon')]\,|Q|$. Therefore, we can find within $D'' \cap Q$ a closed set Γ' whose distance from $\{(t, x): |x| = 1\}$ is not less than $2^{-n-1}(\epsilon^0 - \epsilon')$ and for which $|\Gamma'| \geqslant \epsilon^0 |Q|$. Therefore, with probability not less than δ^0, the process reaches Γ' on $(0, \tau(\partial Q))$. In addition, it is easy to bound from below the probability of passing within a "long" set of type K'', and with it also the probability of reaching Γ from points of Γ' earlier than ∂Q. For the latter probability, only n, $\mu, \nu, \epsilon^0 - \epsilon', \delta^1$ and r enter the estimate. By the same token, for $\epsilon = \epsilon''$ the desired estimate will be obtained with $\delta > 0$, thereby contradicting the assumption. In the second case, a contradiction is obtained even faster, because we can find a set $K = (t, x) + Q(1, R)$ with $8R^2 \geqslant r(\epsilon^0 - \epsilon')$, for which $|K \cap \Gamma| \geqslant q|K|, K \subset Q$.

8. We can show that, under the assumptions of Theorem 1, a similar lower bound (one still depending on R) holds for the expectation of the time taken by the process (t, x_t) in the set Γ up to time $\tau(Q(\beta, R))$. Finally, $\delta \uparrow 1$ when $\epsilon \uparrow 1$ in Theorem 1, and this permits us to prove the theorems of Harnack and Liouville for elliptic and parabolic equations, for example, in the form mentioned in [4], without assuming small dispersion of the eigenvalues.

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