LINEAR ELLIPTIC EQUATIONS OF SECOND ORDER

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ABSTRACT: The classical Schauder type results on $C^{2,\alpha}$-regularity of solutions are exposed for linear second order elliptic equations with Hölder coefficients. Our approach is based on equivalent seminorms in Hölder spaces $C^{k,\alpha}$, which are similar to seminorms introduced by S. Campanato [1]. Under this approach, the $C^{2,\alpha}$-estimates for solutions are derived from the maximum principle and the interior smoothness of harmonic functions.

Contents

1. The Hölder spaces

Let Ω be a domain in n-dimensional Euclidean space R^n, $n \ge 1$. For $k = 0, 1, 2, \cdots$, we denote $C^k(\Omega)$ the set of functions $u = u(x)$ whose derivatives $D^l u$ for $|l| \le k$ are continuous in Ω. Here we use the standard multi-index notation for derivatives, with the understanding $D^0 u = u$. We set

$$|u|_0 = |u|_{0;\Omega} = \sup_{\Omega} |u|, \quad [u]_{k,0} = [u]_{k,0;\Omega} = \max_{|l|=k} |D^l u|_{0;\Omega}. \tag{1.1}$$

Definition 1.1. $C^{k,0}(\Omega)$ is the Banach space of functions $u \in C^k(\Omega)$ with the finite norm

$$|u|_k = |u|_{k,0} = |u|_{k,0;\Omega} = \sum_{j=0}^{k} [u]_{j,0;\Omega}, \quad k = 0, 1, 2, \cdots. \tag{1.2}$$

Further, we call u *Hölder continuous with exponent* α *in* Ω, if the quantity

$$[u]_\alpha = [u]_{\alpha;\Omega} = \sup_{x,y\in\Omega} \frac{|u(x) - u(y)|}{|x - y|^\alpha}, \quad 0 < \alpha \le 1 \tag{1.3}$$

is finite. We set

$$[u]_{k,\alpha} = [u]_{k,\alpha;\Omega} = \max_{|l|=k} [D^l u]_{\alpha;\Omega}. \tag{1.4}$$

Definition 1.2. The Hölder space $C^{k,\alpha}(\Omega)$ is the Banach space of functions $u \in C^k(\Omega)$ with the finite norm

$$|u|_{k,\alpha} = |u|_{k,\alpha;\Omega} = |u|_{k,0;\Omega} + [u]_{k,\alpha;\Omega}, \quad k = 0, 1, 2, \cdots, \quad 0 < \alpha \le 1. \tag{1.5}$$

We will also use the similar notations for closed domains $\overline{\Omega}$. Obviously, for bounded domain Ω we have $C^{k,0}(\overline{\Omega}) = C^k(\overline{\Omega})$. For simplicity we will write $C^{0,\alpha} = C^\alpha$, if $0 < \alpha < 1$.

From the elementary inequality

$$|u(x)v(x) - u(y)v(y)| \le |u(x)| \cdot |v(x) - v(y)| + |v(y)| \cdot |u(x) - u(y)|$$

and (1.1), (1.3) we have

$$[uv]_\alpha \le |u|_0 \cdot [v]_\alpha + |v|_0 \cdot [u]_\alpha, \quad 0 < \alpha \le 1 \tag{1.6}$$

for $u, v \in C^\alpha(\Omega)$. From the definition of derivatives it follows

$$[u]_{k+1,0;\Omega} \le [u]_{k,1;\Omega} \tag{1.7}$$

for $u \in C^{k+1,0}(\Omega),\ k = 0, 1, 2, \cdots$.

The previous inequalities are true for arbitrary domain Ω. In the following statements, we take $\Omega = B_r = B_r(x_0) = \{x \in R^n : |x - x_0| < r\}$ - the ball of radius $r > 0$ centered at $x_0 \in R^n$. Applying the mean value theorem, we get

$$|D^l u(x) - D^l u(y)| \le N|x - y| \cdot [u]_{k+1,0;B_r}$$

for $u \in C^{k+1,0}(B_r)$, $|l| = k$, and $x, y \in B_r$, with a constant N depending only on n. Hence from (1.3), (1.4) it follows

$$[u]_{k,\alpha;B_r} \le N(n) r^{1-\alpha} [u]_{k+1,0;B_r}, \quad k = 0, 1, 2, \cdots, \quad 0 < \alpha \le 1 \tag{1.8}$$

for $u \in C^{k+1,0}(B_r)$.

Lemma 1.1. *Let $u \in C^{k,0}(B_r)$. Then for any ball $B_\rho = B_\rho(x) \subset B_r,\ \rho > 0$, and $|l| = k$, there exists $y \in B_\rho$ such that*

$$|D^l u(y)| \le (2k/\rho)^k |u|_{0;B_r}. \tag{1.9}$$

Proof: We take $h = \rho/k$, $|l| = k$, and consider the difference ratio

$$\delta_h^l u(x) = \delta_{h,1}^{l_1} \delta_{h,2}^{l_2} \cdots \delta_{h,n}^{l_n} u(x), \quad h > 0,$$

where

$$\delta_{h,j} u(x) = \frac{1}{h}[u(x + he_j) - u(x)], \quad e_j \,-\, j\text{-th coordinate vector in } R^n.$$

Then from the mean value theorem we have $\delta_h^l u(x) = D^l u(y)$ for some $y \in B_\rho$, and (1.9) follows from the elementary estimate $|\delta_{h,j} u| \le \frac{2}{h}|u|_0$. **QED**

The following theorem is a particular case ($\Omega = B_r$) of the *interpolation inequalities* (see [2], Sec. 6.8).

Theorem 1.1. *Suppose $j + \beta < k + \alpha$, where $j, k = 0, 1, 2, \cdots$, and $0 \le \alpha, \beta \le 1$. Let $u \in C^{k,\alpha}(B_r)$, where $B_r = B_r(x_0), r > 0$. Then for any $\varepsilon > 0$ we have*

$$r^{j+\beta}[u]_{j,\beta;B_r} \le \varepsilon\, r^{k+\alpha}[u]_{k,\alpha;B_r} + N(\varepsilon)|u|_{0;B_r}, \tag{1.10}$$

with a constant $N(\varepsilon) = N(\varepsilon, n, k, \alpha, \beta)$.

In (1.10) and as a rule henceforth, we omit the dependence of N on the original quantities.

Proof: First of all, by transformation $x \longrightarrow (x - x_0)/r$, the inequality (1.10) is reduced to the equivalent one with $x_0 = 0, r = 1$, so we will deal with $B_r = B_1 = B_1(0)$. For fixed $\varepsilon \in (0, 1]$, we consider different cases:

(a) $j = k,\ 0 = \beta < \alpha$. Let us fix $z \in B_1$, $|l| = k$, and $\rho \in (0, 1)$. For some $x \in B_1,\ \ z \in B_\rho = B_\rho(x) \subset B_1$. By Lemma 1.1, for some $y \in B_\rho$ we have

$$|D^l u(z)| \le |D^l u(z) - D^l u(y)| + |D^l u(y)|$$

$$\le |z - y|^\alpha [u]_{k,\alpha} + (2k/\rho)^k |u|_0 \le (2\rho)^\alpha [u]_{k,\alpha} + (2k/\rho)^k |u|_0.$$

Since $z \in B_1$ and $|l| = k$ can be taken arbitrary, we obtain

$$[u]_{k,0} \le (2\rho)^\alpha [u]_{k,\alpha} + (2k/\rho)^k |u|_0, \quad 0 < \rho < 1. \tag{1.11}$$

Choosing $\rho = \frac{1}{2}\varepsilon^{1/\alpha}$, we arrive at the desired estimate (1.10).

(b) $j = k,\ \ 0 < \beta < \alpha$. By definition (1.4), we can choose $|l| = k$ and $x, y \in B_1$, such that

$$\frac{1}{2}[u]_{k,\beta} \le \frac{|D^l u(x) - D^l u(y)|}{|x - y|^\beta} \le |x - y|^{\alpha - \beta}[u]_{k,\alpha}. \tag{1.12}$$

If $|x-y| \le (\varepsilon/2)^{\frac{1}{\alpha-\beta}}$, we have $[u]_{k,\beta} \le \varepsilon [u]_{k,\alpha}$, and (1.10) is obviously true. Otherwise, the first inequality in (1.12) yields

$$[u]_{k,\beta} \le 4|x-y|^{-\beta}[u]_{k,0} \le N_0[u]_{k,0},$$

where $N_0 = N_0(\varepsilon) = 4(\varepsilon/2)^{\frac{\beta}{\beta-\alpha}}$. From this estimate with $\rho = \frac{1}{2}(\varepsilon/N_0)^{\frac{1}{\alpha}}$ and (1.11) we get (1.10).

(c) $j < k,\ 0 < \alpha$. We can apply (a) with $\varepsilon = 1$ and j in place of k. Then we have

$$[u]_{j,0} \le [u]_{j,1} + N|u|_0.$$

Moreover, for $0 < \beta \le 1$ from (1.8) it follows $[u]_{j,\beta} \le N[u]_{j+1,0}$. Therefore, in any case the estimate

$$[u]_{j,\beta} \le N([u]_{j+1,0} + |u|_0)$$

is true. Iterating this estimate, we obtain

$$[u]_{j,\beta} \le N_0([u]_{k,0} + |u|_0)$$

with a constant $N_0 = N_0(n,k)$. As before, this gives us (1.10).

(d) $\alpha = 0$. From $j+\beta < k+\alpha = k$ it follows $j \le k-1,\ 0 \le \beta < 1$. The previous cases with $\alpha = 1$ bring us to

$$[u]_{j,\beta} \le \varepsilon\, [u]_{k-1,1} + N(\varepsilon)|u|_0 \tag{1.13}$$

for any $\varepsilon > 0$. By virtue of (1.8) $[u]_{k-1,1} \le N[u]_{k,0}$. Putting ε/N in place of ε in (1.13), we complete the proof of theorem. **QED**

Corollary 1.1. *Let $\{u^m\}$, $m = 1, 2, \cdots$, be a bounded sequence in $C^{k,\alpha}(B_r)$, $k = 0, 1, 2, \cdots, 0 < \alpha \le 1$. Suppose that $\{u^m(x)\}$ converges for each $x \in B_r$. Then*

$$u(x) = \lim_{m\to\infty} u^m(x) \in C^{k,\alpha}(B_r), \text{ and } |u|_{k,\alpha} \le A = \sup_m |u^m|_{k,\alpha}. \tag{1.14}$$

Moreover, $\{u^m(x)\}$ converges to u in $C^{j,\beta}(B_r)$, if $j+\beta < k+\alpha$.

Proof: As in the proof of Theorem 1.1, we can assume $r = 1$. By Arzela's theorem $\{u^m\}$ converges to u in $C^0(B_1)$. Let us fix $j+\beta < k+\alpha$ and $\varepsilon_0 > 0$, and then take $\varepsilon = \varepsilon_0/4A$. By (1.10) applied to $u^{m_1} - u^{m_2}$, we have

$$\begin{aligned}[u^{m_1} - u^{m_2}]_{j,\beta} &\le \varepsilon\, [u^{m_1} - u^{m_2}]_{k,\alpha} + N(\varepsilon)|u^{m_1} - u^{m_2}|_0 \\ &\le \frac{\varepsilon_0}{2} + N_0|u^{m_1} - u^{m_2}|_0,\end{aligned}$$

where the constant $N_0 = N_0(\varepsilon_0)$ does not depend on m_1, m_1. Since $\{u^m(x)\}$ is convergent in $C^0(B_1)$, there exists $m_0 = m_0(\varepsilon_0)$ such that

$$N_0|u^{m_1} - u^{m_2}|_0 < \frac{\varepsilon_0}{2} \text{ for } m_1, m_2 > m_0.$$

Therefore,

$$[u^{m_1} - u^{m_2}]_{j,\beta} \leq \varepsilon_0 \text{ for } m_1, m_2 > m_0 = m_0(\varepsilon_0),$$

and $\{u^m(x)\}$ is a Cauchy sequence in $C^{j,\beta}(B_1)$, so it converges to u in $C^{j,\beta}(B_1)$.

In particular, $\{u^m(x)\}$ converges to u in $C^{k,0}(B_1)$. An easy passage to the limit gives us the estimate

$$[u]_{k,\alpha} \leq \sup_m [u^m]_{k,\alpha},$$

which ensures (1.14). **QED**

We will also use special "weighted" spaces $C^{k,\alpha;\gamma}(\Omega)$ for bounded domains $\Omega \subset R^n$. For $k = 0, 1, 2, \cdots$, $0 \leq \alpha \leq 1$, $\gamma \in R^1$, and $u \in C^k(\Omega)$ we set

$$[u]^{(\gamma)}_{k,\alpha} = [u]^{(\gamma)}_{k,\alpha;\Omega} = \sup_{x\in\Omega} d^{k+\alpha+\gamma}(x)[u]_{k,\alpha;B(x)}, \tag{1.15}$$

where

$$d(x) = \frac{1}{2}\text{dist}(x, \partial\Omega), \quad B(x) = B_{d(x)}(x). \tag{1.16}$$

Definition 1.3. For $k = 0, 1, 2, \cdots$, $\gamma \in R^1$, $C^{k;\gamma}(\Omega) = C^{k,0;\gamma}(\Omega)$ is the Banach space of functions $u \in C^k(\Omega)$ with the finite norm

$$\|u\|^{(\gamma)}_{k,0} = \|u\|^{(\gamma)}_{k,0;\Omega} = \sum_{j=0}^{k} [u]^{(\gamma)}_{j,0;\Omega}. \tag{1.17}$$

Definition 1.4. For $k = 0, 1, 2, \cdots$, $0 < \alpha \leq 1$, $\gamma \in R^1$, the *weighted Hölder space* $C^{k,\alpha;\gamma}(\Omega)$ is the Banach space of functions $u \in C^k(\Omega)$ with the finite norm

$$\|u\|^{(\gamma)}_{k,\alpha} = \|u\|^{(\gamma)}_{k,\alpha;\Omega} = \|u\|^{(\gamma)}_{k,0;\Omega} + [u]^{(\gamma)}_{k,\alpha;\Omega}. \tag{1.18}$$

Lemma 1.2. *Let* $\Omega \subset B_{2r}$, $r \geq 1$, *and* $\gamma \geq 0$. *Then* $C^{k,\alpha}(\Omega) \subset C^{k,\alpha;\gamma}(\Omega)$ *and*

$$\|u\|^{(\gamma)}_{k,\alpha;\Omega} \leq r^{k+\alpha+\gamma}|u|_{k,\alpha;\Omega}. \tag{1.19}$$

This lemma follows immediately from the definitions 1.3 and 1.4, because in (1.15) we have $d(x) \leq r$.

Lemma 1.3. *Let* $\beta, \gamma \in R^1$, $0 < \alpha \le 1$, $u \in C^{0,\alpha;\beta}(\Omega)$, *and* $v \in C^{0,\alpha;\gamma}(\Omega)$. *Then*

$$[uv]_{0,\alpha}^{(\gamma+\beta)} \le [u]_{0,0}^{(\beta)} \cdot [v]_{0,\alpha}^{(\gamma)} + [v]_{0,0}^{(\gamma)} \cdot [u]_{0,\alpha}^{(\beta)}, \tag{1.20}$$

$$\|uv\|_{0,\alpha}^{(\gamma+\beta)} \le \|u\|_{0,\alpha}^{(\beta)} \cdot \|v\|_{0,\alpha}^{(\gamma)}. \tag{1.21}$$

Proof: From (1.6) we have

$$d^{\beta+\gamma}[uv]_{\alpha;B} \le d^{\beta}|u|_{0;B} \cdot d^{\gamma}[v]_{\alpha;B} + d^{\gamma}|v|_{0;B} \cdot d^{\beta}[u]_{\alpha;B},$$

where $d = d(x)$, $B = B(x)$ defined in (1.16). Taking the *sup* over $x \in \Omega$, we get (1.20). Analogously, from $|uv|_0 \le |u|_0 \cdot |v|_0$ it follows

$$[uv]_{0,0}^{(\beta+\gamma)} \le [u]_{0,0}^{(\beta)} \cdot [v]_{0,0}^{(\gamma)}.$$

This inequality together with (1.20) yield (1.21). **QED**

By similar reasoning, from (1.8) one can deduce

$$[u]_{k,\alpha}^{(\gamma)} \le N(n)[u]_{k+1,0}^{(\gamma)}, \quad 0 < \alpha \le 1 \tag{1.22}$$

for $u \in C^{k+1,0;\gamma}(\Omega)$. The following equality:

$$\max_{|l|=j}[D^l u]_{k-j,\alpha}^{(\gamma+j)} = [u]_{k,\alpha}^{(\gamma)}, \ \ 0 \le j \le k, \ 0 \le \alpha \le 1 \tag{1.23}$$

for $u \in C^{k,\alpha;\gamma}(\Omega)$ follows from definitions.

Finally, let us put in Theorem 1.1 $r = d = d(x)$, $B_r = B(x)$, multiply both sides of (1.10) by d^{γ}, and then take the *sup* over $x \in \Omega$. As a result we will obtain the following interpolation inequalities for weighted Hölder spaces.

Theorem 1.2. *Suppose* $j + \beta < k + \alpha$, *where* $j, k = 0, 1, 2, \cdots$, *and* $0 \le \alpha, \beta \le 1$. *Let* $u \in C^{k,\alpha;\gamma}(\Omega)$, *where* Ω *is a bounded domain in* R^n *and* $\gamma \in R^1$. *Then for any* $\varepsilon > 0$ *we have*

$$[u]_{j,\beta;\Omega}^{(\gamma)} \le \varepsilon\,[u]_{k,\alpha;\Omega}^{(\gamma)} + N(\varepsilon)|u|_{0,0;\Omega}^{(\gamma)}, \tag{1.24}$$

with a constant $N(\varepsilon) = N(\varepsilon, n, k, \alpha, \beta)$.

Corollary 1.2. *Let* $\{u^m\}$, $m = 1, 2, \cdots$, *be a bounded sequence in* $C^{k,\alpha;\gamma}(\Omega)$, $k = 0, 1, 2, \cdots, 0 < \alpha \le 1$, $\gamma \in R^1$. *Suppose that* $\{u^m(x)\}$ *converges for each* $x \in \Omega$. *Then*

$$u(x) = \lim_{m\to\infty} u^m(x) \in C^{k,\alpha;\gamma}(\Omega), \ \text{ and } \ \|u\|_{k,\alpha}^{(\gamma)} \le A = \sup_m \|u^m\|_{k,\alpha}^{(\gamma)}. \tag{1.25}$$

Proof follows from Corollary 1.1 applied to the balls $B_r = B(x) \subset \Omega$. **QED**

According to (1.15)–(1.17), the norm in $C^{0;\gamma}(\Omega)$ is

$$[u]^{(\gamma)}_{0,0} = \sup_{\Omega} d^{\gamma}(x) \sup_{B(x)} |u|. \tag{1.26}$$

Let us compare this norm with

$$\|u\|^{(\gamma)} = \|u\|^{(\gamma)}_{\Omega} = \sup_{\Omega} d^{\gamma}_x |u(x)|, \quad d_x = \operatorname{dist}(x, \partial\Omega) = 2\, d(x). \tag{1.27}$$

We will use the following obvious relations

$$\frac{1}{2} d_y < d_x < 2\, d_y \quad \text{for all } x \in \Omega,\ y \in B(x) = B_{d(x)}(x). \tag{1.28}$$

Lemma 1.4. *The norms* $|u|^{(\gamma)}_{0,0}$ *and* $\|u\|^{(\gamma)}$ *for* $u \in C^{0;\gamma}(\Omega)$ *are equivalent:*

$$2^{-\gamma} \|u\|^{(\gamma)} \le [u]^{(\gamma)}_{0,0} \le 2^{|\gamma|-\gamma} \|u\|^{(\gamma)}. \tag{1.29}$$

Proof: The first inequality in (1.29) follows from

$$2^{-\gamma} \cdot d^{\gamma}_x |u(x)| = d^{\gamma}(x) |u(x)| \le d^{\gamma}(x) \sup_{B(x)} |u|.$$

Further, (1.27) and (1.28) yield

$$|u(y)| \le d_y^{-\gamma} \|u\|^{(\gamma)} \le 2^{|\gamma|} d_x^{-\gamma} \|u\|^{(\gamma)} = 2^{|\gamma|-\gamma} d^{-\gamma}(x) \|u\|^{(\gamma)}$$

for all $y \in B(x)$, that gives us the second inequality. **QED**

Remark 1.1. In the standard approach to the Schauder interior estimates (see [2], Ch.6) the notation $[u]^{(\gamma)}_{k,\alpha}$ is used for

$$A = \max_{|l|=k} \sup_{x,y \in \Omega} d^{k+\alpha+\gamma}_{x,y} \frac{|D^l u(x) - D^l u(y)|}{|x-y|^{\alpha}} = \sup_{\delta > 0} \delta^{k+\alpha+\gamma} [u]_{k,\alpha;\Omega_\delta}, \tag{1.30}$$

where $0 < \alpha \le 1$, $k + \alpha + \gamma \ge 0$, $d_{x,y} = \min(d_x, d_y)$, and

$$\Omega_\delta = \{x \in \Omega : d_x = \operatorname{dist}(x, \partial\Omega) > \delta\}. \tag{1.31}$$

One can show that for Lipschitz domains Ω seminorms $[u]^{(\gamma)}_{k,\alpha}$ in (1.15) and (1.30) are equivalent, if $k + \alpha + \gamma \ge 0$. In particular,

$$N^{-1} [u]_{k,\alpha} \le [u]^{(-k-\alpha)}_{k,\alpha;\Omega} \le [u]_{k,\alpha} \tag{1.32}$$

with a constant N depending only on k, α and Ω.

However, in the case $k + \alpha + \gamma < 0$ we have $A < \infty$ in (1.30) only for polynomials of degree at most k (and then $A = 0$), while $[u]^{(\gamma)}_{k,\alpha} < \infty$ for more general class of functions. For example, if $k + \alpha + \gamma < 0 \le k + 1 + \gamma$ and $u \in C^{k+1}(\overline{B_1})$, then by virtue of (1.22) we have

$$[u]^{(\gamma)}_{k,\alpha} \le N [u]^{(\gamma)}_{k+1,0} \le N [u]_{k+1} < \infty. \tag{1.33}$$

2. On equivalent seminorms in Hölder spaces

Let Ω be a bounded domain in R^n, $k = 0, 1, 2, \cdots$, $0 < \alpha \leq 1$, and $\gamma \in R^1$. We denote $\mathcal{P}_k$ the collection of all polynomials of degree at most k. The *Taylor polynomial* of degree k for the function u at the point $y \in R^n$ is

$$T_{y,k}\, u(x) = \sum_{|l| \leq k} \frac{D^l u(y)}{l!} (x-y)^l \in \mathcal{P}_k. \tag{2.1}$$

Lemma 2.1. *Let $u \in C^{k,\alpha}(\overline{\Omega})$, $0 < \alpha \leq 1$. Then for any $x, y \in \overline{\Omega}$ such that the segment $[x,y] \subset \overline{\Omega}$, we have*

$$|u(x) - T_{y,k}\, u(x)| \leq N(n)[u]_{k,\alpha} \cdot |x-y|^{k+\alpha}. \tag{2.2}$$

Proof: By Taylor's formula,

$$u(x) = T_{y,k-1}\, u(x) + \sum_{|l|=k} \frac{D^l u(\xi)}{l!} (x-y)^l,$$

where $\xi \in [x,y]$. Therefore,

$$|u(x) - T_{y,k}\, u(x)| = \left| \sum_{|l|=k} \frac{D^l u(\xi) - D^l u(y)}{l!} (x-y)^l \right|$$

$$\leq N(n) \max_{|l|=k} |D^l u(\xi) - D^l u(y)| \cdot |x-y|^k.$$

Finally, from (1.4) it follows

$$\max_{|l|=k} |D^l u(\xi) - D^l u(y)| \leq [u]_{k,\alpha} \cdot |\xi - y|^\alpha \leq [u]_{k,\alpha} \cdot |x-y|^\alpha,$$

that completes the proof. **QED**

Corollary 2.1. *Let $u \in C^{k,\alpha}(B_\rho)$, $B_\rho = B_\rho(x_0)$. Then*

$$E_k[u; B_\rho] = \inf_{p \in \mathcal{P}_k} \sup_{B_\rho} |u - p| \leq N(n)[u]_{k,\alpha}\, \rho^{k+\alpha}. \tag{2.3}$$

Lemma 2.2. *Let $k = 0, 1, 2, \cdots$, $0 < \alpha \leq 1$, and $u \in C^{k,\alpha}(B_\rho)$, $B_\rho = B_\rho(x_0)$. Then for any $\varepsilon > 0$ we have*

$$\rho^{-\alpha} \max_{|l|=k} \operatorname*{osc}_{B_\rho} D^l u \;\leq\; \varepsilon [u]_{k,\alpha;B_\rho} + N(\varepsilon) \rho^{-k-\alpha} E_k[u; B_\rho] \tag{2.4}$$

with a constant $N(\varepsilon) = N(\varepsilon, n, k, \alpha)$, where $\operatorname{osc} f = \sup f - \inf f$.

Proof: Using the elementary inequality $\operatorname{osc} f \le 2\sup|f|$ and (1.10) with $r=\rho$, $j=k$, $\beta=0$, we have

$$\frac{1}{2}\rho^{-\alpha}\max_{|l|=k}\operatorname*{osc}_{B_\rho} D^l u \;\le\; \rho^{-\alpha}[u]_{k,0;B_\rho} \;\le\; \varepsilon[u]_{k,\alpha;B_\rho} + N(\varepsilon)\rho^{-k-\alpha}\sup_{B_\rho}|u|.$$

For arbitrary $p \in \mathcal{P}_k$, the left-hand side of this inequality and $[u]_{k,\alpha}$ remain the same if we replace u by $u-p$. After the replacement, we take the infimum of the right-hand side over $p \in \mathcal{P}_k$. On redefining ε, this will give us the desired estimate. **QED**

The next theorem is similar to Theorem 2.1 in [3] (see also [1]).

Theorem 2.1. *Let $k=0,1,2,\cdots$, $0<\alpha\le 1$, $\gamma\in R^1$, and $u\in C^k(\Omega)$ has a finite seminorm $[u]^{(\gamma)}_{k,\alpha}$ in (1.15). Set*

$$M^{(\gamma)}_{k,\alpha} = M^{(\gamma)}_{k,\alpha}[u;\Omega] = \sup_{x\in\Omega} d^{k+\alpha+\gamma}(x) \sup_{\rho\in(0,d(x)]} \rho^{-k-\alpha}E_k[u;B_\rho(x)], \tag{2.5}$$

where $d(x)=\frac{1}{2}\operatorname{dist}(x,\partial\Omega)$, E_k is defined in (2.3). Then seminorms $[u]^{(\gamma)}_{k,\alpha}$ and $M^{(\gamma)}_{k,\alpha}$ are equivalent :

$$N_1^{-1}[u]^{(\gamma)}_{k,\alpha} \le M^{(\gamma)}_{k,\alpha} \le N_2[u]^{(\gamma)}_{k,\alpha} \tag{2.6}$$

with some constants $N_1 = N_1(n,k,\alpha,\gamma)$ and $N_2 = N_2(n)$.

Proof: By virtue of Corollary 2.1, for all $x\in\Omega$ and $\rho\in(0,d(x)]$

$$d^{k+\alpha+\gamma}(x)\rho^{-k-\alpha}E_k[u;B_\rho(x)] \le N d^{k+\alpha+\gamma}(x)[u]_{k,\alpha;B_{\rho(x)}} \le N[u]^{(\gamma)}_{k,\alpha;\Omega},$$

that gives us the second inequality in (2.6).

To prove the first inequality, we fix $x_0\in\Omega$, $d=d(x_0)=\frac{1}{2}\operatorname{dist}(x_0,\partial\Omega)$, $|l|=m$, and $x,y\in B_d(x_0)$, such that

$$A = [u]^{(\gamma)}_{k,\alpha} \le 2d^{k+\alpha+\gamma}\frac{|D^l u(x)-D^l u(y)|}{|x-y|^\alpha}. \tag{2.7}$$

We consider separately the cases (a) $\rho=|x-y|<d/2$ and (b) $\rho\ge d/2$. In the case (a), x,y lie in a ball $B_\rho(z)\subset B(x_0)=B_d(x_0)$. Since $d=d(x_0)\le 2d(z)$, from (2.7) it follows

$$A \le N_0\, d^{k+\alpha+\gamma}(z)\cdot\rho^{-\alpha}\operatorname*{osc}_{B_\rho(z)} D^l u \tag{2.8}$$

with $N_0=N_0(k,\alpha,\gamma)$. Obviously, this inequality is also true in the case (b) for $z=x_0,\rho=d$, and $N_0=2^{1+\alpha}$. In any case, we have (2.8), where $0<\rho\le d(z)$.

Using Lemma 2.2 and the definition of A and $M_{k,\alpha}^{(\gamma)}$, we now obtain for any $\varepsilon > 0$

$$A \le N_0\varepsilon \cdot d^{k+\alpha+\gamma}(z)[u]_{k,\alpha;B(z)} + N(\varepsilon)d^{k+\alpha+\gamma}(z) \cdot \rho^{-k-\alpha}E_k[u; B_\rho(z)]$$

$$\le N_0\varepsilon \cdot A + N(\varepsilon)M_{k,\alpha}^{(\gamma)}.$$

For $\varepsilon = 1/2N_0$, we get the desired estimate $A \le N_1 M_{k,\alpha}^{(\gamma)}$. **QED**

Theorem 2.2. *Let $k = 0, 1, 2, \cdots$, $0 < \alpha \le 1$, and $u \in C^k(R^n)$ has a finite seminorm $[u]_{k,\alpha;R^n}$ in (1.4). Set*

$$M_{k,\alpha} = M_{k,\alpha}[u; R^n] = \sup_{x\in R^n, \rho>0} \rho^{-k-\alpha}E_k[u; B_\rho(x)]. \tag{2.9}$$

Then seminorms $[u]_{k,\alpha}$ and $M_{k,\alpha}$ are equivalent :

$$N_1^{-1}[u]_{k,\alpha} \le M_{k,\alpha} \le N_2[u]_{k,\alpha} \tag{2.10}$$

with some constants $N_1 = N_1(n)$ and $N_2 = N_2(n, k, \alpha)$.

Proof of this theorem, with some simplifications, is analogous to the previous one.

3. The maximum principle

In this section we will study some simple properties of *linear elliptic operators*

$$Lu = \sum_{i,j=1}^{n} a_{ij}D_{ij}u + \sum_{i=1}^{n} b_i D_i u + cu, \tag{3.1}$$

defined for $u \in C^2(\Omega)$. Here $D_i u = \partial u/\partial x_i$, $D_{ij}u = D_i D_j u$, and a_{ij}, b_i, c are given functions on a domain $\Omega \subset R^n$. We impose the following

Assumptions 3.1. (a) (Uniform ellipticity condition). There exists a constant $\nu \in (0, 1]$ such that

$$\nu|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \le \nu^{-1}|\xi|^2 \tag{3.2}$$

in Ω for all $\xi \in R^n$.

(b) There exists a constant $K \ge 0$ such that

$$\sum_{i=1}^{n} |b_i(x)| \le K, \quad c(x) \le 0 \tag{3.3}$$

for all $x \in \Omega$.

From linear algebra it is known that for any symmetric matrix $A = [a_{ij}]$ there exists an orthogonal matrix P such that $P^*AP = \Lambda$, where $P^* = P$ inverse $= P^{-1}$, and Λ is a diagonal matrix whose diagonal elements are the eigenvalues $\lambda_1, \cdots, \lambda_n$ of A. Since the orthogonal matrix preserves the dot product in R^n, for $\eta = P^*\xi$ we have $\xi = P\eta$, $|\xi| = |\eta|$, and

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j = \xi^* A\xi = \eta^* P^* AP\eta = \eta^* \Lambda\eta = \sum_{k=1}^{n} \lambda_k \eta_k^2.$$

Therefore, (3.2) is true for all $\xi \in R^n$ if and only if $\lambda_k \in [\nu, \nu^{-1}]$ for all $k = 1, 2, \cdots, n$. Using these facts, one can show that from (3.2) it follows

$$a_{ii} \geq \nu, \quad |a_{ij}| \leq \nu^{-1}, \quad i, j = 1, 2, \cdots, n, \tag{3.4}$$

$$\sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq \sum_{i=1}^{n} a_{ii} \cdot |\xi|^2, \quad \xi \in R^n. \tag{3.5}$$

Further, the equality $P^*AP = \Lambda$ yields $A = P\Lambda P^*$, and

$$a_{ij} = \sum_{k=1}^{n} \lambda_k \xi_i^k \xi_j^k, \tag{3.6}$$

where $\xi_i, \cdots, \xi_n$ are columns of P^*.

Remark 3.1. It is also possible to derive (3.4), (3.5) directly from (3.2). Indeed, putting $\xi_i = 1$ for fixed i, and $\xi_j = 0$ for $j \neq i$, we get $\nu \leq a_{ii} \leq \nu^{-1}$. Furhter, taking $\xi_j, -\xi_i$ in place of ξ_i, ξ_j for fixed $i \neq j$, and $\xi_k = 0$ for $k \neq i, j$, we have

$$a_{ii}\xi_j^2 + a_{jj}\xi_i^2 - 2a_{ij}\xi_i\xi_j \geq 0, \quad i, j = 1, 2, \cdots, n.$$

Summation over all i, j brings us to (3.5). Finally, selecting $\xi_i, \xi_j = \pm 1$, we obtain $|a_{ij}| \leq \frac{1}{2}(a_{ii} + a_{jj}) \leq \nu^{-1}$.

Lemma 3.1. *Let constants $\nu \in (0, 1]$, $K \geq 0$, and $r > 0$ be fixed. Then there exists a function $v_0 \in C^\infty(\overline{B_r})$, $B_r = B_r(0)$, such that*

$$Lv_0 \leq -1 \quad \text{in} \quad B_r \tag{3.7}$$

for any operator L of the form (3.1) with coefficients satisfying Assumptions 3.1, where $\Omega = B_r$. Moreover,

$$0 < v_0 \leq N_0 = N_0(\nu, K, r) \quad \text{in } B_r, \quad v_0 = 0 \quad \text{on } \partial B_r. \tag{3.8}$$

Proof: Let us consider the function $\cosh(\lambda|x|)$, $\lambda > 0$. This function belongs to C^∞, because the Taylor expansion of $\cosh t$ contains only even powers of t. We have

$$D_i \cosh(\lambda|x|) = \lambda \sinh(\lambda|x|)\xi_i, \quad \text{where } \xi = |x|^{-1}x,$$
$$D_{ij} \cosh(\lambda|x|) = \lambda^2 \cosh(\lambda|x|)\xi_i\xi_j + \lambda|x|^{-1}\sinh(\lambda|x|)(\delta_{ij} - \xi_i\xi_j).$$

Having in mind that $|\xi| = 1$, $\sinh t < \cosh t$, and using (3.2)–(3.5), we obtain:

$$\begin{aligned}(L-c)\cosh(\lambda|x|) &= \left(\sum a_{ij}D_{ij} + \sum b_i D_i\right)\cosh(\lambda|x|)\\ &\geq \lambda^2\cosh(\lambda|x|)\sum a_{ij}\xi_i\xi_j + \lambda\sinh(\lambda|x|)\sum b_i\xi_i\\ &\geq \cosh(\lambda|x|)(\lambda^2\nu - \lambda K) \geq \lambda(\lambda\nu - K) \geq 1\end{aligned}$$

under appropriate choice of $\lambda = \lambda(\nu, K) > 0$.

Now we take $v_0(x) = \cosh(\lambda r) - \cosh(\lambda|x|)$. Then (3.8) is obviouisly true with $N_0 = \cosh(\lambda r)$. Moreover, since $c \leq 0$, we have

$$Lv_0 \leq (L-c)v_0 = -(L-c)\cosh(\lambda|x|) \leq -1 \quad \text{in } B_r,$$

so (3.7) is also true. **QED**

The following theorem is known as the *weak maximum principle* (see [2], Sec. 3.1).

Theorem 3.1. *Let coefficients of L satisfy Assumptions 3.1, and $c = 0$ in a bounded domain $\Omega \subset R^n$. Suppose that a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \geq 0$ $(Lu \leq 0)$ in Ω. Then*

$$\sup_\Omega u = \sup_{\partial\Omega} u \quad (\inf_\Omega u = \inf_{\partial\Omega} u). \tag{3.9}$$

Proof: First we show that if $Lu > 0$, then u cannot achieve an interior maximum in Ω. Indeed, for such a point x_0, $D_i u(x_0) = 0$ for all i, and

$$\sum_{i,j=1}^n D_{ij}u(x_0)\xi_i\xi_j = \frac{d^2}{dt^2}u(x_0 + t\xi) \leq 0$$

for all $\xi \in R^n$. By virtue of (3.6), $Lu(x_0) = \sum a_{ij}(x_0)D_{ij}u(x_0) \leq 0$, contradicting $Lu > 0$.

In the case $Lu \geq 0$, we take the function v_0 in Lemma 3.1 for a ball $B_r \supset \Omega$. Then for any $\varepsilon > 0$, $L(u - \varepsilon v_0) \geq \varepsilon > 0$, so the previous considerations give us

$$\sup_\Omega (u - \varepsilon v_0) = \sup_{\partial\Omega}(u - \varepsilon v_0).$$

Letting $\varepsilon \to 0$, we obtain $\sup_\Omega u = \sup_{\partial\Omega} u$. **QED**

Under the assumptions $Lu \geq 0$, $c \leq 0$, this theorem can be applied to $L_0 u = Lu - cu \geq -cu \geq 0$ in $\Omega^+ = \{u > 0\} \subset \Omega$. Hence for $u^\pm = \max\{\pm u, 0\}$ we obtain:

Corollary 3.1. *Let $c \le 0$ in Ω. Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $Lu \ge 0$ $(Lu \le 0)$ in Ω. Then*

$$\sup_{\Omega} u^+ = \sup_{\partial\Omega} u^+ \quad (\sup_{\Omega} u^- = \sup_{\partial\Omega} u^-). \tag{3.10}$$

Applying this corollary to $u - v$, we get the *comparison principle*:

Theorem 3.2. *Let $c \le 0$ in Ω. Suppose that $u, v \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfy $Lu \ge Lv$ in Ω, $u \le v$ on $\partial\Omega$. Then $u \le v$ in Ω. If $Lu = Lv$ in Ω, $u = v$ on $\partial\Omega$, then $u = v$ in Ω.*

Theorem 3.3. *Let $Lu = f$, $c \le 0$ in a bounded domain $\Omega \subset R^n$, where $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Then*

$$\sup_{\Omega} |u| \le \sup_{\partial\Omega} |u| + N_0 \cdot \sup_{\Omega} |f| \tag{3.11}$$

with a constant $N_0 = N_0(\nu, K, \operatorname{diam}\Omega)$.

Proof: We have $\Omega \subset B_r$ for some ball $B_r = B_r(x_0)$, $r = \operatorname{diam}\Omega$. Consider the function

$$v(x) = \sup_{\partial\Omega} |u| + \sup_{\Omega} |f| \cdot v_0(x - x_0),$$

where v_0 is a function in Lemma 3.1. By virtue of (3.7) and (3.8),

$$Lv(x) \le \sup_{\Omega} |f| \cdot Lv_0(x - x_0) \le -\sup_{\Omega} |f| \le \pm f(x) = \pm Lu(x)$$

in Ω, and $v \ge |u|$ on $\partial\Omega$. By the previous theorem, $|u| \le v$ in Ω, and (3.11) is true with constant N_0 in (3.8). **QED**

In the following definition of sub- and supersolutions, we do not rely on the existence of solutions of the equation $Lu = f$.

Definition 3.1. A function $w \in C(\Omega)$ is a *subsolution (supersolution)* of the equation $Lu = f$ in Ω if for every ball $B \subset \overline{B} \subset \Omega$ and every function $v \in C^2(B) \cap C(\overline{B})$ such that $Lv < f$ $(Lv > f)$ in B, the inequality $v \ge w$ $(v \le w)$ on ∂B implies also $v > w$ $(v < w)$ in B.

Remark 3.2. Obviously, w is a subsolution of $Lu = f$ if and only if $-w$ is a supersolution of $Lu = -f$, so the properties of sub- and supersolutions are quite similar. For certainty we will consider subsolutions. If $a_{ij}, b_i, c, f \in C(\Omega)$, then any subsolution $w \in C^2(\Omega)$ satisfies $Lw \ge f$ in Ω, because otherwise we have $Lw < f$ in a ball $B \subset \overline{B} \subset \Omega$, and the choice $v = w$ contradicts the definition of subsolution. Moreover, the inverse statement is true without assumption of continuity of a_{ij}, b_i, c and f: if a function

$w \in C^2(\Omega)$ satisfies $Lw \geq f$ in Ω, then it is a subsolution of $Lu = f$ in Ω. Indeed, for $v \in C^2(B) \cap C(\overline{B})$ such that $Lv < f$ in $B \subset \overline{B} \subset \Omega$ and $v \geq w$ on ∂B, we have $L(w-v) > 0$ in B. Therefore, as in the proof of Theorem 3.1, $w - v$ cannot achieve an interior maximum:

$$w - v < \sup_{\partial B}(w - v) \leq 0 \quad \text{in } B.$$

Lemma 3.2. *Let $w \in C(\overline{\Omega})$ be a subsolution of $Lu = f$ in Ω. Then for every $v \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $Lv < f$ in Ω, the inequality $v \geq w$ on $\partial\Omega$ implies $v > w$ in Ω. If $Lv \leq f$ in Ω, then $v \geq w$ on $\partial\Omega$ implies $v \geq w$ in Ω.*

The corresponding statement can be made for supersolutions.

Proof: Let $Lv < f$ in Ω, $v \geq w$ on $\partial\Omega$, but the inequality $v > w$ in Ω is not true. Then

$$0 \geq -\mu = \inf_{\Omega}(v - w) = v(y) - w(y) \ \text{ for some } y \in \Omega. \tag{3.12}$$

Choosing a ball $B_r = B_r(y) \subset \overline{B_r(y)} \subset \Omega$, we have

$$L(v + \mu) \leq Lv < f \ \text{ in } B_r, \quad v + \mu \geq w \ \text{ on } \partial B_r.$$

By definition of subsolution w, $v + \mu > w$ in B_r, so $v(y) + \mu > w(y)$. This contradiction with (3.12) proves inequality $v > w$ in Ω.

If $Lv \leq f$ in Ω, we take v_0 in Lemma 3.1 for a ball $B_r \supset \Omega$, and then $v^\varepsilon = v + \varepsilon v_0$, $\varepsilon > 0$ satisfy

$$Lv^\varepsilon \leq f - \varepsilon < f \ \text{ in } \Omega, \quad v^\varepsilon \geq v \geq w \ \text{ on } \partial\Omega.$$

Therefore, $v^\varepsilon > w$, and $v = \lim_{\varepsilon \to 0} v^\varepsilon \geq w$ in Ω. **QED**

Theorem 3.4. *Let $w \in C(\Omega)$ be given. Suppose that for every $y \in \Omega$ there exists a subsolution w^y of $Lu = f$ in a ball B^y such that*

$$y \in B^y \subset \overline{B^y} \subset \Omega, \quad w^y \leq w \ \text{ in } B^y, \quad w^y(y) = w(y). \tag{3.13}$$

Then w is a subsolution of $Lu = f$ in Ω.

Proof: Suppose the statement of theorem is not true. Then there exist a ball $B \subset \overline{B} \subset \Omega$ and a function $v \in C^2(B) \cap C(\overline{B})$ such that $Lv < f$ in B, $v \geq w$ on ∂B, but the inequality $v > w$ fails in B. Hence

$$0 \geq -\mu = \inf_{B}(v - w) = v(y) - w(y) \ \text{ for some } y \in B. \tag{3.14}$$

Let us fix a ball $B_r = B_r(y) \subset \overline{B_r(y)} \subset B \cap B^y$. We have

$$L(v + \mu) \leq Lv < f \ \text{ in } B_r, \quad v + \mu \geq w \geq w^y \ \text{ on } \partial B_r,$$

and $v + \mu > w^y$ in B_r, because w^y is a subsolution of $Lu = f$ in $B^y \supset \overline{B_r}$. In particular, $v(y) + \mu > w^y(y) = w(y)$, contradicting (3.14). **QED**

Corollary 3.2. *Let w_1 and w_2 be subsolutions of $Lu = f$ in domains Ω_1 and Ω_2 corerspondingly. Suppose that*

$$w_1 \ge w_2 \;\text{ on } \Omega_1 \cap \partial\Omega_2, \quad w_2 \ge w_1 \;\text{ on } \Omega_2 \cap \partial\Omega_1. \tag{3.15}$$

Then the function w defined on $\Omega = \Omega_1 \cap \Omega_2$ by

$$w(x) = \begin{cases} w_1(x), & x \in \Omega_1 \setminus \Omega_2, \\ w_2(x), & x \in \Omega_2 \setminus \Omega_1, \\ \max\{w_1(x), w_2(x)\}, & x \in \Omega_1 \cap \Omega_2, \end{cases} \tag{3.16}$$

belongs to $C(\Omega)$, and w is a subsolution of $Lu = f$ in Ω.

Using this corollary, we will construct some special subsolutions of

$$Lu = d^{\beta-2} \;\text{ in } \Omega, \quad \text{where } d = d_x = \operatorname{dist}(x, \partial\Omega), \; 0 < \beta < 1. \tag{3.17}$$

First we consider the case $\Omega = B_R = B_R(0)$ for fixed $R > 0$. In this case $d = R - |x|$, $x \in B_R$.

Lemma 3.3. *Let constants $0 < \beta < 1$ and $R > 0$ be fixed. Then for $\Omega = B_R = B_R(0)$ and any operator L under Assumptions 3.1 on coefficients, there exists a subsolution $w \in C(\overline{\Omega})$ of the equation (3.17) such that*

$$0 \ge w \ge -N_1 d^\beta \tag{3.18}$$

in Ω with a constant $N_1 = N_1(\nu, K, \beta, R)$.

Proof: We have: $d = R - |x|, \quad D_i d^\beta = -\beta d^{\beta-1}\xi_i,$

$$D_{ij} d^\beta = \beta(\beta-1) d^{\beta-2}\xi_i\xi_j + \beta|x|^{-1} d^{\beta-1}(\xi_i\xi_j - \delta_{ij}),$$

where $\xi = |x|^{-1}x$. Since $|\xi| = 1$, by virtue of (3.2)–(3.5)

$$Ld^\beta \le \left(\sum a_{ij}D_{ij} + \sum b_i D_i\right) d^\beta$$

$$\le \beta(\beta-1)d^{\beta-2}\sum a_{ij}\xi_i\xi_j - \beta d^{\beta-1}\sum b_i\xi_i \le \beta d^{\beta-2}[(\beta-1)\nu + Kd].$$

Since $\beta(1-\beta) < 0$, we can choose constants $\beta_0 > 0$ and $\delta_0 \in (0, R/2)$ depending only on ν, K, β, R, such that

$$Ld^\beta < -\beta_0 d^{\beta-2}, \quad 0 < d < 2\delta_0. \tag{3.19}$$

Furhter, by Lemma 3.1 there exists a function $v_0 \in C^\infty(\overline{B_R})$ such that

$$Lv_0 \le -1, \quad 0 \le v_0 \le N_0 \tag{3.20}$$

in $\Omega = B_R$ with a constant $N_0 = N_0(\nu, K, R)$. We set

$$w_1 = -N_1 d^\beta, \quad w_2 = -N_1\delta_0^\beta - \delta_0^{\beta-2} v_0, \tag{3.21}$$

where $N_1 = \max\{\beta_0^{-1}, (2^\beta - 1)^{-1}\delta_0^{-2} v_0\}$.

Let us represent $\Omega = B_R$ in the form

$$\Omega = \Omega_1 \cup \Omega_2, \quad \text{where} \quad \Omega_1 = \Omega \cap \{d < 2\delta_0\}, \ \ \Omega_2 = \Omega \cap \{d > \delta_0\}. \tag{3.22}$$

From (3.19)–(3.22) it follows

$$Lw_1 \geq N_1\beta_0 d^{\beta-2} \geq d^{\beta-2} \ \text{ in } \ \Omega_1, \quad Lw_2 \geq \delta_0^{\beta-2} \geq d^{\beta-2} \ \text{ in } \ \Omega_2,$$

$$w_1 - w_2 = \delta_0^{\beta-2} v_0 \geq 0 \quad \text{ on } \quad \Omega_1 \cap \partial\Omega_2 = \{d = \delta_0\},$$

$$w_2 - w_1 \geq N_1(2^\beta - 1)\delta_0^\beta - \delta_0^{\beta-2} v_0 \geq \delta_0^{\beta-2}(N_0 - v_0) \geq 0 \quad \text{on} \quad \Omega \setminus \Omega_1.$$

So w_1 and w_2 satisfy all the assumptions of Corollary 3.2 with $f = d^{\beta-2}$, and the function w defined in (3.16) is a subsolution of $Lu = d^{\beta-2}$. Finally, since $w_2 \geq w_1$ on $\Omega \setminus \Omega_1$, the inequalities (3.18) are also true: $0 \geq w \geq w_1 \geq -N_1 d^\beta$ in $\Omega = B_R$. **QED**

Now we will extend the statement of Lemma 3.3 to more general domains Ω satisfying the *exterior sphere condition*:

Assumptions 3.2. For every point $y^* \in \partial\Omega$, there exists a ball $B = B_R(z)$ satisfying $\overline{B} \cap \overline{\Omega} = y^*$, where the constant $R > 0$ does not depend on y^*.

Theorem 3.5. *The statement of Lemma 3.3 is extended to bounded domains Ω satisfying Assumptions 3.2, with a constant $N_1 = N_1(\nu, K, \beta, R, \operatorname{diam}\Omega)$ in (3.18).*

Proof: From Assumptions 3.2 it follows

$$d = d_x = \operatorname{dist}(x, \partial\Omega) = \min_{z \in Z}(|x - z| - R), \quad x \in \Omega,$$

where $Z = \{z \in R^n : \operatorname{dist}(x, \Omega) = R\}$. Denoting $h(x) = |x| - R$, we get

$$d^\beta = d_x^\beta = \min_{z \in Z} h^\beta(x - z). \tag{3.23}$$

Quite analogously to (3.19), one can prove

$$Lh^\beta < -\beta_0 h^{\beta-2}, \quad 0 < h < 2\delta_0 \tag{3.24}$$

with constants $\beta_0, \delta_0 > 0$ depending only on ν, K, β, R.

For fixed $y \in \Omega_1 = \Omega \cap \{d < 2\delta_0\}$, we take $z \in Z$ satisfying $h(y-z) = d_y$, and set $w^y(x) = -h^\beta(x-z)$. Then (3.24) is equivalent to

$$Lw^y(x) > \beta_0 d_x^{\beta-2} \tag{3.25}$$

at the point $x = y$. By continuity, (3.25) remains true in a ball B^y such that $y \in B^y \subset \overline{B^y} \subset \Omega_1$. Moreover,

$$w^y \le -d^\beta \quad \text{in} \ \ B^y, \qquad w^y(y) = -d_y^\beta.$$

Applying Theorem 3.4, we see that $-d^\beta$ is a subsolution of $Lu = \beta_0 d^{\beta-2}$ in Ω_1. We can use this fact in place of (3.19) in the proof of Lemma 3.3. The next relations (3.20) are true in a ball $B_r \supset \Omega$, $r = \operatorname{diam}\Omega$, and for the rest, the proof is the same. **QED**

The following statement is similar to Lemma 6.21 in [2].

Corollary 3.3. *Under assumptions of Theorem 3.5, let $u \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of $Lu = f$ in Ω, $u = 0$ on $\partial\Omega$. Then*

$$\|u\|_\Omega^{(-\beta)} \le N_1 \, \|f\|_\Omega^{(2-\beta)}, \tag{3.26}$$

where $0 < \beta < 1$, $N_1 = N_1(\nu, K, \beta, R, \operatorname{diam}\Omega)$ is a constant in (3.18), and norms $\|\cdot\|^{(\gamma)}$ are defined in (1.27).

Proof: We set

$$A = \|f\|^{(2-\beta)} = \sup d_x^{2-\beta} |f(x)|.$$

Since w is a subsolution of $Lu = d^{\beta-2}$, and $Ad^{\beta-2} \ge f$, we obtain that Aw is a subsolution of $Lu = f$ in Ω. We also have $u = w = 0$ on $\partial\Omega$, hence by Lemma 3.2 and (3.18), $u \ge Aw \ge -N_1 A d^\beta$ in Ω.

Obviously, the same inequalities are true for $-u$ in place of u. Therefore,

$$|u| \le N_1 A d^\beta \quad \text{in} \ \ \Omega, \quad \|u\|^{(-\beta)} = \sup_\Omega d^{-\beta}|u| \le N_1 A,$$

that completes the proof. **QED**

4. Some estimates for harmonic functions

Let Ω be a domain in R^n, and $u \in C^2(\Omega)$. The function u is called *harmonic (subharmonic, superharmonic)* in Ω if it satisfies

$$\Delta u = \sum_{i=1}^n D_{ii} u = 0 \quad (\ge 0, \ \le 0) \quad \text{in} \ \ \Omega. \tag{4.1}$$

In other words (see Remark 3.2), sub- and superharmonic functions in $C^2(\Omega)$ are sub- and supersolutions of the *Laplace equation* $\Delta u = 0$. Since $\Delta u = \operatorname{div} Du = 0$ in Ω, from the divergence theorem we have

$$\int_{\partial\Omega_0} \frac{\partial u}{\partial \nu}\, dS = \int_{\partial\Omega_0} Du \cdot \nu\, dS = \int_{\Omega_0} \Delta u\, dx = 0 \tag{4.2}$$

for any subdomain $\Omega_0 \subset \overline{\Omega_0} \subset \Omega$ with smooth boundary $\partial\Omega_0$, where dS is the $(n-1)$-dimensional area element on $\partial\Omega_0$, and ν is the unit outward normal to $\partial\Omega_0$. As a consequence of (4.2), we will derive the *mean value theorem*:

Theorem 4.1. *Let $u \in C^2(\Omega)$ satisfy $\Delta u = 0$ in Ω. Then for any ball $B = B_r(x_0) \subset \overline{B} \subset \Omega$, we have*

$$u(x_0) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B} u\, dS, \tag{4.3}$$

where $\omega_n = 2\pi^{n/2}/\Gamma(n/2)$ is the surface area of the unit sphere in R^n.

Proof: Without loss of generality we assume $x_0 = 0$. Consider the function

$$I(\rho) = \rho^{1-n} \int_{\partial B_\rho} u(x)\, dS_x = \int_{\partial B_1} u(\rho y)\, dS_y, \quad 0 < \rho \le r.$$

Since $u(\rho y)$ is harmonic inB_1, and for $y \in \partial B_1$ the outward normal $\nu = y$, by virtue of (4.2) we get

$$\frac{dI(\rho)}{d\rho} = \int_{\partial B_1} Du(\rho y) \cdot y\, dS_y = \int_{\partial B_1} \frac{\partial u(\rho y)}{\partial \nu}\, dS_y = 0.$$

Therefore, $I(\rho) = \text{const}$, $0 < \rho \le r$, and

$$I(r) = \lim_{\rho \to 0+} I(\rho) = \omega_n \cdot u(0),$$

that gives us (4.3). **QED**

We will prove the smoothness of harmonic functions, using only the mean value theorem and some general properties of functions. We fix a function $\zeta(x) \in C^\infty(R^n)$ depending only on $|x|$, such that

$$\zeta \ge 0 \text{ on } R^n, \quad \zeta = 0 \text{ on } R^n \setminus B_1(0), \quad \text{and} \quad \int_{R^n} \zeta(x)\, dx = 1. \tag{4.4}$$

For $u \in \mathcal{L}^1_{loc}(\Omega)$ and $\varepsilon > 0$, the *regularization* of u, denoted by $u^{(\varepsilon)}$, is defined by the equality

$$u^{(\varepsilon)}(x) = \int_{B_1(0)} u(x - \varepsilon y)\zeta(y)\, dy = \varepsilon^{-n} \int_\Omega u(z)\zeta(\varepsilon^{-1}(x - z))\, dz \tag{4.5}$$

provided $x \in \Omega_\varepsilon = \{x \in \Omega : d_x = \operatorname{dist}(x, \partial\Omega) > \varepsilon\}$.

Lemma 4.1. *(a) For arbitrary* $\varepsilon > 0$*, we have* $u^{(\varepsilon)} \in C^\infty(\Omega_\varepsilon)$*, and*

$$[u^{(\varepsilon)}]_{k,0;\Omega_\varepsilon} = \max_{|l|=k} \sup_{\Omega_\varepsilon} |D^l u^{(\varepsilon)}| \le N\varepsilon^{-k} \sup_\Omega |u|, \quad k = 0, 1, 2, \cdots \tag{4.6}$$

with a constant $N = N(n, k)$*.*

(b) Suppose $|u(x) - u(y)| \le \omega(\varepsilon)$ *for all* $x, y \in \Omega$ *such that* $|x - y| \le \varepsilon$*. Then*

$$\sup_{\Omega_\varepsilon} |u^{(\varepsilon)} - u| \le \omega(\varepsilon). \tag{4.7}$$

In particular,

$$\sup_{\Omega_\varepsilon} |u^{(\varepsilon)} - u| \le \varepsilon^\alpha \, [u]_{\alpha;\Omega}, \quad 0 < \alpha \le 1. \tag{4.8}$$

(c) If $u \in C^k(\Omega)$*, then* $D^l u^{(\varepsilon)} = (D^l u)^{(\varepsilon)}$ *on* Ω_ε *for all* $|l| \le k$*. Moreover,*

$$|u^{(\varepsilon)}|_{k,\alpha;\Omega_\varepsilon} \le |u|_{k,\alpha;\Omega}, \quad k = 0, 1, 2, \cdots, 0 \le \alpha \le 1. \tag{4.9}$$

In (4.8) and (4.9), we use notations (1.1)–(1.4).

Proof: All these statements follow directly from (4.5) and corresponding definitions. Indeed, for $|l| = k$, $x \in \Omega$ we have

$$D^l u^{(\varepsilon)}(x) = \varepsilon^{-n-k} \int_{B_\varepsilon(x)} u(z) D^l \zeta(\varepsilon^{-1}(x - z)) \, dz,$$

$$|D^l u^{(\varepsilon)}(x)| \le \varepsilon^{-n-k} \mathrm{mes}\, B_\varepsilon \cdot [\zeta]_{k,0} \cdot \sup_\Omega |u|,$$

so (4.6) is true with $N = \mathrm{mes}\, B_1 \cdot [\zeta]_{k,0}$.

Further, (4.7) follows from

$$u^{(\varepsilon)}(x) - u(x) = \int_{B_1(0)} [u(x - \varepsilon y) - u(x)] \zeta(y) \, dy, \quad x \in \Omega_\varepsilon,$$

and finally, (4.9) is a consequence of the first equality in (4.5). **QED**

Lemma 4.2. *If* u *is harmonic in a domain* $\Omega \subset R^n$*, then* $u = u^{(\varepsilon)}$ *on* $\Omega_\varepsilon = \{x \in \Omega : d_x = \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$*,* $\varepsilon > 0$*. Moreover,* $u \in C^\infty(\Omega)$*, and*

$$\max_{|l|=k} \sup_\Omega d_x^k |D^l u| \le N \cdot \sup_\Omega |u|, \quad k = 0, 1, 2, \cdots \tag{4.10}$$

with a constant $N = N(n, k)$*.*

Using notations (1.15), (1.16) for weighted seminorms, we can rewrite (4.10) in the form

$$[u]^{(0)}_{k,0;\Omega} \le N \cdot \sup_\Omega |u|, \quad k = 0, 1, 2, \cdots. \tag{4.11}$$

Proof: We have

$$u^{(\varepsilon)}(x) = \int_{B_1(0)} u(x-\varepsilon y)\zeta(y)\,dy = \int_0^1 dr \int_{|y|=r} u(x-\varepsilon y)\zeta(y)\,dS_y \qquad (4.12)$$

for $x \in \Omega_\varepsilon$. Since $\zeta(y) = const$ on the sphere $\{|y| = r\} = \partial B_r(0)$, by virtue of (4.3)

$$\int_{|y|=r} u(x-\varepsilon y)\,dS_y = u(x)\cdot\omega_n r^{n-1} = u(x)\int_{|y|=r} dS_y.$$

Therefore, from (4.5) we get

$$u^{(\varepsilon)}(x) = u(x)\int_0^1 dr \int_{|y|=r} \zeta(y)\,dS_y = u(x)\int_{B_1(0)} \zeta(y)\,dy = u(x), \quad x \in \Omega_\varepsilon.$$

Finally, (4.10) follows from (4.6) with $\varepsilon = d_x = \mathrm{dist}(x, \partial\Omega)$. **QED**

The rest part of this section is devoted to the *simplest* linear elliptic operators of the form (3.1) which are slightly more general than the Laplace operator Δ. Namely, will consider operalors L^0 with coefficients satisfying the following

Assumptions 4.1. $a_{ij} = const$ and satisfy the uniform ellipticity condition (3.2) with a constant $\nu \in (0,1]$, $b_i = 0$, $c = 0$.

In other words, $L^0 u = \sum a_{ij} D_{ij} u$, $a_{ij} = const$. Such operators are reduced to Δ by a nonsingular linear transformation from R^n onto R^n. First we consider the *Dirichlet problem* in unit ball:

$$L^0 u = \sum_{i,j=1}^n a_{ij} D_{ij} u = f \ \text{ in } \ B_r = B_r(x_0), \quad u = \varphi \ \text{ on } \ \partial B_r \qquad (4.13)$$

with *polynomials* f and φ.

Lemma 4.3. *Let polynomials $f \in \mathcal{P}_k$ and $\varphi \in \mathcal{P}_{k+2}$ be given. Then under Assumptions 4.1 on L^0, there exists a unique solution of the problem (4.13) in $C^2(B_1) \cap C(\overline{B_1})$. This solution is represented in the form $u = \varphi + (r^2 - |x|^2)g$, where $g \in \mathcal{P}_k$.*

Proof: Without loss of generality we can assume $r = 1, x_0 = 0$. Uniqueness of solution is contained in Theorem 3.2. To prove the existence, we set $u = v + \varphi$, so (4.13) is reduced to the equivalent problem

$$L^0 v = f_0 \ \text{ in } \ B_1, \quad v = 0 \ \text{ on } \ \partial B_1 \qquad (4.14)$$

with $f_0 = f - L^0\varphi \in \mathcal{P}_k$. Let us consider the linear mapping

$$Tp = L^0((1-|x|^2)p): \ \mathcal{P}_k \longrightarrow \mathcal{P}_k.$$

If $Tp = 0$, then $u = (1-|x|^2)p$ is a solution of the problem (4.13) with $f \equiv 0$, $\varphi \equiv 0$. By uniqueness, then $u \equiv 0$, $p \equiv 0$. Hence T has the null space $T^{-1}(0) = \{0\}$. Since the dimension of $\mathcal{P}_k$ is finite, we have $T(\mathcal{P}_k) = \mathcal{P}_k$. Then $Tg = f_0$ for some $g \in \mathcal{P}_k$, and $v = (1-|x|^2)g$ is a solution of (4.14). **QED**

The following lemma will be useful for extension of the existence results for the problem (4.13) from the case when f and φ are polynomials to more general cases.

Lemma 4.4. *Let $0 < \alpha \le 1$, $\gamma \in R^1$, and $\{u^m\}$, $m = 1, 2, \cdots$, be a bounded sequence in $C^{2,\alpha;\gamma}(\Omega)$, i.e. $\|u^m\|_{2,\alpha}^{(\gamma)} \le A = const < \infty$ for all m. Suppose that there exist the limits*

$$u(x) = \lim_{m\to\infty} u^m(x), \quad f(x) = \lim_{m\to\infty} Lu^m(x) \quad \text{for all } x \in \Omega$$

with $L = \sum a_{ij}(x)D_{ij} + \sum b_i(x)D_i + c(x)$. Then $u(x) \in C^{2,\alpha;\gamma}(\Omega)$, $\|u\|_{2,\alpha}^{(\gamma)} \le A$, and $Lu = f$ in Ω.

Proof: Let us fix $x \in \Omega$, $d = \frac{1}{2}\operatorname{dist}(x, \partial\Omega)$, and $B = B_d(x)$. Since $\{u^m\}$ is bounded in $C^{2,\alpha;\gamma}(\Omega)$, it is also bounded in $C^{2,\alpha}(B)$ and convergent to u on B. By Corollary 1.1, $u \in C^{2,\alpha}(B)$, and $u^m \to u$ in $C^2(B)$, hence

$$Lu = \lim_{m\to\infty} Lu^m = f \quad \text{on } \Omega.$$

The other statements of lemma are contained in Corollary 1.2. **QED**

Theorem 4.2. *Let $B_r = B_r(x_0)$ and a function $\varphi \in C(\overline{B_r})$ be given. Then under Assumptions 4.1 on L^0, the Dirichlet problem*

$$L^0 u = \sum_{i,j=1}^{n} a_{ij} D_{ij} u = 0 \ \text{ in } B_r, \quad u = \varphi \ \text{ on } \partial B_r \tag{4.15}$$

has a unique solution in $C^\infty(B_r) \cap C(\overline{B_r})$. Moreover, the estimates (4.10), (4.11) are true, where $\Omega = B_r$, $N = N(n, \nu, k)$.

Proof: Relying on the Weierstrass approximation theorem, we choose a sequence of polynomials $\{\varphi_m\}$ such that $|\varphi - \varphi_m| \le 1/m$ on $\overline{B_r}$, $m = 1, 2, \cdots$. By the previous lemma, there exist a sequence of polynomials $\{u_m\}$ satisfying

$$L^0 u_m = 0 \ \text{ in } B_r, \quad u_m = \varphi_m \ \text{ on } \partial B_r. \tag{4.16}$$

From the comparison principle we have

$$\sup_{B_r} |u_i - u_j| \le \sup_{\partial B_r} |\varphi_i - \varphi_j| \le \frac{1}{i} + \frac{1}{j} \to 0 \quad \text{as } i, j \to \infty.$$

Therefore $\{u_m\}$ is a Cauchy sequence in $C(\overline{B_r})$, so it is convergent to a function $u \in C(\overline{B_r})$.

Further, there exists a nonsingular linear transformation $y = Ax$ converting the equation $L^0u(x) = 0$ in B_r into $\Delta u(y) = 0$ in $\Omega = A(B_r)$. Then the estimates (4.10), (4.11) for $u(y)$ in Ω produce the analogous estimates for $u(x)$ in B_r with $N = N(n, \nu, k)$. Hence we have such estimates for each u^m with N independent on m. An easy limit passage shows that $u = \lim u_m \in C^\infty(B_r)$ and u satisfies the same estimates. Finally, Lemma 4.4 with $\gamma = 0, \alpha = 1$ gives us the equality $L^0u = \lim L^0u^m = 0$. **QED**

Corollary 4.1. *Let $r > 0$ and $x_0 \in R^n_0 = \{x \in R^n : x_n = 0\}$ be given. Set*

$$B_r^+ = B_r^+(x_0) = \{x \in B_r(x_0) : x_n > 0\}. \tag{4.17}$$

(a) For any function $\varphi \in C(\overline{B_r^+})$ satisfying the equality $\varphi = 0$ on $\Gamma = R^n_0 \cap B_r(x_0)$, the Dirichlet problem

$$\Delta u = 0 \ \text{ in } \ B_r^+, \quad u = \varphi \ \text{ on } \ \partial B_r^+ \tag{4.18}$$

has a unique solution $u \in C^\infty(B_r^+ \cup \Gamma) \cap C(\overline{B_r^+})$. Moreover,

$$[u]_{k,0;B^+_{r/2}} = \max_{|l|=k} \sup_{B^+_{r/2}} |D^l u| \le N(n,k) r^{-k} \sup_{B_r^+} |u|, \quad k = 0, 1, 2, \cdots. \tag{4.19}$$

(b) For arbitrary function $\varphi \in C(\overline{B_r^+})$, the equation $\Delta u = 0$ in B_r^+, with the boundary conditions

$$D_n u = 0 \ \text{ on } \ \Gamma, \quad u = \varphi \ \text{ on } \ \partial B_r^+ \setminus \Gamma, \tag{4.20}$$

has a unique solution $u \in C^\infty(B_r^+ \cup \Gamma) \cap C(\overline{B_r^+})$, and the estimate (4.19) holds.

Proof: (a) We take the odd continuation of φ from $\partial B_r^+ \setminus \Gamma$ to ∂B_r:

$$\varphi(x_1, \cdots, x_{n-1}, -x_n) = -\varphi(x_1, \cdots, x_{n-1}, x_n).$$

Then by uniqueness the solution of the problem

$$\Delta u = 0 \ \text{ in } \ B_r, \quad u = \varphi \ \text{ on } \ \partial B_r$$

satisfies the similar equality and, therefore, (4.18), (4.19) hold.

For the proof of (b), we take the even continuation:

$$\varphi(x_1, \cdots, x_{n-1}, -x_n) = \varphi(x_1, \cdots, x_{n-1}, x_n),$$

that gives us (4.19), (4.20). **QED**

5. The Schauder interior estimates

Let Ω be a domain in R^n. As in Section 3, we consider linear operators

$$Lu = \sum_{i,j=1}^{n} a_{ij} D_{ij} u + \sum_{i=1}^{n} b_i D_i u + cu \tag{5.1}$$

for $u \in C^2(\Omega)$. Now we need some smoothness of coefficients in terms of weighted Hölder spaces $C^{k,\alpha;\gamma}(\Omega)$ defined in (1.17), (1.18).

Assumptions 5.1. For some constants $0 < \alpha < 1$ and $K_1 > 0$, we have

$$\max_{i,j} \|a_{ij}\|_{0,\alpha}^{(0)}, \quad \max_{i} \|b_i\|_{0,\alpha}^{(1)}, \quad \|c\|_{0,\alpha}^{(2)} \quad \le \quad K_1, \tag{5.2}$$

where norms $\|u\|_{0,\alpha}^{(\gamma)}$ for function on Ω are defined in (1.18).

Theorem 5.1. *Let coefficients of operator L in (5.1) satisfy Assumptions 5.1 with some constants* $0 < \alpha < 1$ *and* $K_1 > 0$. *Then for arbitrary* $\gamma \in R^1$ *and* $u \in C^{2,\alpha;\gamma}(\Omega)$, *we have* $f = Lu \in C^{0,\alpha;\gamma+2}(\Omega)$, *and*

$$\|f\|_{0,\alpha}^{(\gamma+2)} \le N_1 \cdot \|u\|_{2,\alpha}^{(\gamma)} \tag{5.3}$$

with a constant $N_1 = N_1(n, K_1)$.

Proof: We will use the inequality (1.21) in Lemma 1.3 and the estimate

$$\|D^l u\|_{k-j,\alpha}^{(\gamma+j)} \le \|u\|_{k,\alpha}^{(\gamma)}, \quad 0 \le j \le k, \; 0 \le \alpha \le 1, \tag{5.4}$$

which obviously follows from (1.23). Together with (5.2), this gives us

$$\|a_{ij} D_{ij} u\|_{0,\alpha}^{(\gamma+2)} \le \|a_{ij}\|_{0,\alpha}^{(0)} \cdot \|D_{ij} u\|_{0,\alpha}^{(\gamma+2)} \le K_1 \cdot \|u\|_{2,\alpha}^{(\gamma)}$$

for all i, j. Analogously, having in mind also (1.22), we obtain

$$\|b_i D_i u\|_{0,\alpha}^{(\gamma+2)} \le \|b_i\|_{0,\alpha}^{(1)} \cdot \|D_i u\|_{0,\alpha}^{(\gamma+1)} \le K_1 \cdot \|u\|_{1,\alpha}^{(\gamma)} \le N \cdot \|u\|_{2,0}^{(\gamma)},$$

$$\|cu\|_{0,\alpha}^{(\gamma+2)} \le \|c\|_{0,\alpha}^{(2)} \cdot \|u\|_{0,\alpha}^{(\gamma)} \le K_1 \cdot \|u\|_{0,\alpha}^{(\gamma)} \le N \cdot \|u\|_{1,0}^{(\gamma)}$$

with $N = N(n, K_1)$. These estimates yield the desired estimate (5.3) for $f = \sum a_{ij} D_{ij} u + \sum b_i D_i u + cu$. **QED**

Remark 5.1. Notice that for lower order terms we have

$$\|\sum b_i D_i u + cu\|_{0,\alpha}^{(\gamma+2)} \le N(n, K_1) \cdot \|u\|_{2,0}^{(\gamma)}. \tag{5.5}$$

In the following theorem we show that for elliptic operators L with Hölder coefficients the norms of functions u and $f = Lu$ in (5.3) are "almost" equivalent.

Theorem 5.2. *Let coefficients of operator L in (5.1) satisfy both Assumptions 3.1 and 5.1 with some constants* $\nu, \alpha \in (0,1)$ *and* $K, K_1 \geq 0$. *Then for arbitrary* $\gamma \in R^1$ *and* $u \in C^{2,\alpha;\gamma}(\Omega)$, *we have*

$$\|u\|_{2,\alpha}^{(\gamma)} \leq N \cdot \left([u]_{0,0}^{(\gamma)} + [f]_{0,\alpha}^{(\gamma+2)}\right) \tag{5.6}$$

with a constant $N = N(n, \nu, K, K_1, \alpha, \gamma)$, *where* $f = Lu$.

Proof: Throughout the proof we will denote by N different constants depending only on n, ν, K, K_1, α, and γ. We will also use the brief notations

$$U_{2,\alpha} = [u]_{2,\alpha}^{(\gamma)}, \quad U_k = [u]_{k,0}^{(\gamma)}, \quad F_\alpha = [f]_{0,\alpha}^{(\gamma+2)}, \quad F_0 = [f]_{0,0}^{(\gamma+2)}. \tag{5.7}$$

Step 1. Let $\gamma \in R^1$ and $u \in C^{2,\alpha;\gamma}(\Omega)$ be given. We first assume $b_i = 0$, $c = 0$ in Ω. Let us fix

$$y \in \Omega, \quad d = d(y) = \frac{1}{2}\,\mathrm{dist}\{y, \partial\Omega\}, \quad \rho \in (0, d], \quad \text{and} \quad \varepsilon \in (0, 1/2].$$

We set $r = \rho/\varepsilon$ and consider separately the cases (a) $r \leq d$ and (b) $r > d$.

In the case (a), we take

$$a_{ij}^0 = a_{ij}(y), \quad L^0 = \sum_{i,j=1}^n a_{ij}^0 D_{ij}, \quad \varphi = u - T_{y,2}u,$$

and define v as the solution of the problem

$$L^0 v = \sum a_{ij}^0 D_{ij} v = 0 \ \text{ in } \ B_r = B_r(y), \quad u = \varphi \ \text{ on } \ \partial B_r. \tag{5.8}$$

From Theorem 4.2 it follows $v \in C^\infty(B_r) \cap C(\overline{B_r})$ and

$$[v]_{3,0;B_{r/2}} \leq N r^{-3} \sup_{B_r} |v| = N r^{-3} \sup_{\partial B_r} |\varphi|.$$

Having in mind that $\rho = \varepsilon r \leq r/2$, applying Corollary 2.1 to v in B_ρ, and then Lemma 2.1 to u in B_r, we obtain (with different constants N):

$$\rho^{-2-\alpha} E_2[v; B_\rho] \leq N \rho^{1-\alpha} [v]_{3,0;B_{r/2}} \leq N \rho^{1-\alpha} r^{-3} \sup_{B_r} |\varphi|$$

$$\leq N \rho^{1-\alpha} r^{\alpha-1} [u]_{2,\alpha;B_r} = N \varepsilon^{1-\alpha} [u]_{2,\alpha;B_r}.$$

Since $r \leq d$, by definition of $U_{2,\alpha} = [u]_{2,\alpha}^{(\gamma)}$ in (1.15) we get

$$d^{2+\alpha+\gamma} \rho^{-2-\alpha} E_2[v; B_\rho] \leq N \varepsilon^{1-\alpha} U_{2,\alpha}. \tag{5.9}$$

Step 2. Now we proceed to evaluate $\varphi - v$ on B_r. We have

$$L^0(\varphi - v) = L^0\varphi = \sum a^0_{ij}[D_{ij}u(x) - D_{ij}u(y)].$$

By our assumption $b_i = 0, c = 0$, we have $f = \sum a_{ij}D_{ij}u$, therefore, the previous equality can be rewritten as follows:

$$L^0(\varphi - v)(x) = \sum[a^0_{ij} - a_{ij}(x)]D_{ij}u(x) + f(x) - f(y). \tag{5.10}$$

Further, by virtue of (5.2) and definition of seminorms (5.7), for each $x \in B_r = B_r(y) \subset B_d$ we have

$$|a^0_{ij} - a_{ij}(x)| = |a_{ij}(y) - a_{ij}(x)| \le r^\alpha [a_{ij}]_{\alpha;B_r} \le d^{-\alpha} r^\alpha K_1,$$

$$|D_{ij}u(x)| \le [u]_{2,0;B_r} \le d^{-2-\gamma} U_2,$$

$$|f(x) - f(y)| \le r^\alpha [f]_{\alpha;B_r} \le d^{-2-\alpha-\gamma} r^\alpha F_\alpha.$$

Therefore, from (5.10) it follows

$$|L^0(\varphi - v)(x)| \le Ar^\alpha \quad \text{in } \ B_r, \tag{5.11}$$

where

$$A = d^{-2-\alpha-\gamma}(n^2 K_1 U_2 + F_\alpha). \tag{5.12}$$

Notice that functions $\varphi - v$ together with

$$w(x) = \frac{Ar^\alpha}{2n\nu}(r^2 - |x-y|^2)$$

satisfy the relations

$$L^0 w \le -Ar^\alpha \le -|L^0(\varphi - v)| \ \text{ in } \ B_r = B_r(y), \quad w = \varphi - v = 0 \ \text{ on } \ \partial B_r.$$

By the comparison principle we get

$$\sup_{B_\rho} |\varphi - v| \le \sup_{B_r} |\varphi - v| \le \sup_{B_r} |w| = \frac{A}{2n\nu} r^{2+\alpha}.$$

Using the equality $r = \rho/\varepsilon$ and (5.12), we obtain the estimate

$$d^{2+\alpha+\gamma}\rho^{-2-\alpha} \sup_{B_\rho} |\varphi - v| \le N\varepsilon^{-2-\alpha}(U_2 + F_\alpha). \tag{5.13}$$

Step 3. Now we will combine together (5.9) and (5.13). Obviously

$$E_2[u; B_\rho] \le E_2[v; B_\rho] + E_2[\varphi - v; B_\rho] \le E_2[v; B_\rho] + \sup_{B_\rho} |\varphi - v|,$$

so we receive

$$d^{2+\alpha+\gamma}\rho^{-2-\alpha}E_2[u;B_\rho] \le N\varepsilon^{1-\alpha}U_{2,\alpha} + N\varepsilon^{-2-\alpha}(U_2+F_\alpha). \tag{5.14}$$

We have considered the case (a) $r=\rho/\varepsilon\le d$. In the case (b) $r=\rho/\varepsilon>d$, we have $d^{2+\alpha}\rho^{-2-\alpha}=\varepsilon^{-2-\alpha}$, and

$$d^\gamma E_2[u;B_\rho] \le d^\gamma \sup_{B_d}|u| \le U_2,$$

so the left hand side of (5.14) does not exceed $N\varepsilon^{-2-\alpha}U_0$. Since $y\in\Omega$ and $0<\rho\le d=d(y)$ are chosen in an arbitrary manner, we get the following estimate for the seminorm in Theorem 2.1:

$$M_{2,\alpha}^{(\gamma)} \le N\varepsilon^{1-\alpha}U_{2,\alpha} + N\varepsilon^{-2-\alpha}(U_2+U_1+U_0+F_\alpha) \tag{5.15}$$

for all $\varepsilon>0$. By this theorem, (5.15) remains valid with $U_{2,\alpha}$ in place of $M_{2,\alpha}^{(\gamma)}$. Choosing then $\varepsilon=\varepsilon(n,\nu,K,K_1,\alpha,\gamma)>0$ such that the coefficient of $U_{2,\alpha}$ would be less then $1/2$, we get

$$U_{2,\alpha} \le N\cdot(U_2+U_1+U_0+F_\alpha). \tag{5.16}$$

Step 4. We have proved (5.16) under the additional restrictions $b_i=0$, $c=0$. To cover the general case, it suffices to rewrite $Lu=f$ in the form

$$\sum a_{ij}D_{ij}u = f_0 = f - \sum b_iD_iu - cu,$$

and then use Remark 5.1, so we get

$$[f_0]_{0,\alpha}^{(\gamma+2)} \le [f]_{0,\alpha}^{(\gamma+2)} + N\cdot(U_2+U_1+U_0).$$

Finally, from (5.16) and the interpolation inequalities (Theorem 1.2)

$$U_2+U_1 \le \varepsilon\, U_{2,\alpha} + N(\varepsilon)U_0, \quad \varepsilon>0,$$

it follows

$$\|u\|_{2,\alpha}^{(\gamma)} = U_{2,\alpha}+U_2+U_1+U_0 \le N\cdot(U_0+F_\alpha),$$

completing the proof of theorem. **QED**

Theorem 5.3. *In addition to the assumptions of Theorem 5.2, suppose that the domain Ω is bounded and satisfies Assumptions 3.2 (the exterior sphere condition) with a constant $R>0$. Then for arbitrary $\beta\in(0,1)$ and $u\in C^{2,\alpha;-\beta}(\Omega)$, we have $f=Lu\in C^{0,\alpha;2-\beta}(\Omega)$, and*

$$N_1^{-1}\cdot\|f\|_{0,\alpha}^{(2-\beta)} \le \|u\|_{2,\alpha}^{(-\beta)} \le N_2\cdot\|f\|_{0,\alpha}^{(2-\beta)} \tag{5.17}$$

with a constant $N_1=N_1(n,K_1)$ and $N_2=N_2(n,\nu,K,K_1,\alpha,\beta,R,\operatorname{diam}\Omega)$.

Proof: The first inequality in (5.17) is contained in Theorem 5.1. To prove the second inequality we observe that by Lemma 1.4, the norms $[u]_{0,0}^{(-\beta)}$ and $\|u\|^{(-\beta)}$ are equivalent, so the Corollary 3.3 gives us the estimate

$$\|u\|^{(-\beta)} \le N \cdot \|f\|^{(2-\beta)}.$$

Applying then Theorem 5.2 with $\gamma = -\beta$, we have

$$\|u\|_{2,\alpha}^{(-\beta)} \le N \cdot ([f]_{0,0}^{(2-\beta)} + [f]_{0,\alpha}^{(2-\beta)}) = N \cdot \|f\|_{0,\alpha}^{(2-\beta)},$$

and the secont inequality in (5.17) is also true. **QED**

Remark 5.2. The inequalities (5.17) show that the linear mapping

$$L \ : \ \ u \in C^{2,\alpha;-\beta}(\Omega) \quad \longrightarrow \quad f = Lu \in C^{0,\alpha;2-\beta}(\Omega)$$

if bounded and, moreover, the inverse mapping $u = L^{-1}f$ is bounded on the set $L(C^{2,\alpha;-\beta}) \subset C^{0,\alpha;2-\beta}$. Since $0 < \beta < 1$, each function $u \in C^{2,\alpha;-\beta}$ vanishes on $\partial\Omega$, so that it can be considered as the solution of the Dirichlet problem

$$Lu = f \ \text{ in } \ \Omega, \qquad u = 0 \ \text{ on } \ \partial\Omega. \tag{5.18}$$

Therefore, the coincidence

$$L(C^{2,\alpha;-\beta}) = C^{0,\alpha;2-\beta} \tag{5.19}$$

implies the solvability of the problem (5.18) for each $f \in C^\alpha(\Omega) \subset C^{0,\alpha;2-\beta}(\Omega)$ in the class $C^{2,\alpha;-\beta}(\Omega) \subset C^{2,\alpha}_{loc}(\Omega) \cap C(\overline{\Omega})$.

In the next section, we will prove the coincidence (5.19) for $\Omega = B_r$, and then in Section 7 we will extend this result to Ω satisfying Assumptions 3.2.

6. The Dirichlet problem in a ball

Throughout this section a ball $B_r = B_r(x_0) \subset R^n$ and constants $\alpha, \beta \in (0,1)$ are fixed. We will consider the Dirichlet problem

$$Lu = f \ \text{ in } \ B_r, \qquad u = \varphi \ \text{ on } \ \partial B_r \tag{6.1}$$

under Assumptions 3.1 and 5.1 on the coefficients of the linear elliptic operator $L = \sum a_{ij}D_{ij} + \sum b_iD_i + c$. We start with the case $\varphi = 0$, and then we will pass to $\varphi \in C(\overline{B_r})$.

Theorem 6.1. *Under the above conditions, for each* $f \in \mathcal{B}_2 = C^{0,\alpha;2-\beta}(B_r)$ *there exists a unique* $u \in \mathcal{B}_1 = C^{2,\alpha;-\beta}(B_r)$ *satisfying* $Lu = f$ *in* B_r. *In other words,* L *maps* $\mathcal{B}_1$ *onto* $\mathcal{B}_2 :$ $\ L(\mathcal{B}_1) = \mathcal{B}_2$.

As was mentioned in Remark 5.2, the functions $u \in \mathcal{B}_1$ vanish on ∂B_r, so we have $u \in C(\overline{B_r})$, $u = 0$ on ∂B_r, and this theorem provides the solution of the problem (6.1) with $f \in \mathcal{B}_2 \supset C^{\alpha}(B_r)$, $\varphi = 0$.

Proof: Uniqueness follows from the comparison principle. To prove the existence, we denote for shortness $\|\cdot\|_1$ and $\|\cdot\|_2$ norms in the Banach spaces $\mathcal{B}_1$ and $\mathcal{B}_2$, so that (5.17) has the form

$$N_1^{-1} \cdot \|f\|_2 \le \|u\|_1 \le N_2 \cdot \|f\|_2, \quad f = Lu. \tag{6.2}$$

Step 1. $L = \Delta$, $f \in C^1(\overline{B_r})$. By the Weierstrass theorem, there exists a sequence of polynomials $\{f^m\}$ convergent to f in $C^1(\overline{B_r})$. For each m, Lemma 4.3 guarantees the existence of $u^m \in \mathcal{B}_1$ satisfying $\Delta u^m = f^m$. Since $2-\beta > 0$, the convergence in C^1 implies the convergence in $\mathcal{B}_2 = C^{0,\alpha;2-\beta}(B_r)$. Hence from (6.2) it follows

$$\|u^m - u^k\|_1 \le N_2 \cdot \|f^m - f^k\|_2 \to 0 \text{ as } m,k \to \infty.$$

In other words, $\{u^m\}$ is a Cauchy sequence in $\mathcal{B}_1$, so it is convergent to a function u satisfying $\Delta u = f$.

Step 2. $L = \Delta$, $f \in C^{\alpha}(B_r)$, and $f = 0$ in a neighborhood of ∂B_r. If $\varepsilon > 0$ is small, the regularization $f^{(\varepsilon)}$ in (4.5) is well defined. By Step 1, there exists $u^{\varepsilon} \in \mathcal{B}_1$ satisfying $\Delta u^{\varepsilon} = f^{(\varepsilon)}$. From (6.2) and (4.9) we obtain the uniform boundedness of $\{u^{\varepsilon}\}$ in $\mathcal{B}_1$:

$$\|u^{\varepsilon}\|_1 \le N_2 \cdot \|f^{(\varepsilon)}\|_2 \le N \cdot |f^{(\varepsilon)}|_{\alpha;B_{r-\varepsilon}} \le N \cdot |f|_{\alpha;B_r}.$$

Moreover, by virtue of (4.8) we have $f^{(\varepsilon)} \to f$ in $C(\overline{B_r})$, and then the comparison principle (Theorem 3.3) gives us

$$\sup_{B_r} |u^{\varepsilon_1} - u^{\varepsilon_2}| \le N_0 \cdot \sup_{B_r} |f^{(\varepsilon_1)} - f^{(\varepsilon_2)}| \to 0 \quad \text{as } \varepsilon_1, \varepsilon_2 \to 0.$$

Therefore, there exists $\lim_{\varepsilon \to 0} u^{\varepsilon} = u \in C(\overline{B_r})$. Applying Lemma 4.4, we get $u \in \mathcal{B}_1$, and $\Delta u = \lim \Delta u^{\varepsilon} = \lim f^{(\varepsilon)} = f$ in B_r.

Step 3. $L = \Delta$, $f \in \mathcal{B}_2$. For small $\varepsilon > 0$, let us take auxiliary functions $\eta^{\varepsilon} \in C^{\infty}(B_r)$ satisfying the following conditions:

$$\eta^{\varepsilon} = 0 \text{ on } B_r \setminus B_{r-\varepsilon}, \quad \eta^{\varepsilon} = 0 \text{ on } B_{r-3\varepsilon}, \quad \text{and } |D\eta^{\varepsilon}| \le N/\varepsilon \tag{6.3}$$

with a constant N independent on ε. It is easy to see that (6.3) holds for the regularization $\eta^{\varepsilon} = h_{\varepsilon}^{(\varepsilon)}$ of the function $h_{\varepsilon}:\ \ h_{\varepsilon} = 1$ on $B_{r-2\varepsilon}$, and $h_{\varepsilon} = 0$ elsewhere. From (1.22), (6.3), and definition of weighted norms we have

$$\|\eta^{\varepsilon}\|_{0,\alpha}^{(0)} \le N \cdot \|\eta^{\varepsilon}\|_{1,0}^{(0)} \le N(n), \quad \varepsilon > 0. \tag{6.4}$$

Further, the functions $f^\varepsilon = \eta^\varepsilon f$ satisfy all the conditions of Step 2. Moreover, the inequalities (1.21) and (6.4) yield

$$\|f^\varepsilon\|_2 = \|\eta^\varepsilon f\|_{0,\alpha}^{(2-\beta)} \le \|\eta^\varepsilon\|_{0,\alpha}^{(0)} \cdot \|f\|_{0,\alpha}^{(2-\beta)} \le N \cdot \|f\|_{(2)}$$

for small $\varepsilon > 0$. By (6.2), the solutions u^ε of $\Delta u^\varepsilon = f^\varepsilon$ are bounded in $\mathcal{B}_1$. In turn, this gives us the estimate

$$|u^\varepsilon(x)| \le (r - |x|)^\beta \|u^\varepsilon\|_{0,0}^{(-\beta)} \le (r - |x|)^\beta \|u^\varepsilon\|_2 \le N \cdot (r - |x|)^\beta \qquad (6.5)$$

for all $x \in B_r$ and small $\varepsilon > 0$.

To prove the convergence of u^ε as $\varepsilon \to 0$, we observe that

$$\Delta(u^\varepsilon - u^{\varepsilon'}) = (\eta^\varepsilon - \eta^{\varepsilon'})f = 0 \quad \text{on } B_{r-3\varepsilon} \quad \text{for } 0 < \varepsilon' < \varepsilon.$$

By the maximum principle and (6.5),

$$\sup_{B_r} |u^\varepsilon - u^{\varepsilon'}| = \sup_{B_r \setminus B_{r-3\varepsilon}} |u^\varepsilon - u^{\varepsilon'}| \le N\varepsilon^\beta, \quad 0 < \varepsilon' < \varepsilon.$$

Thus functions u^ε are bounded in $\mathcal{B}_1$ and convergent in $C(\overline{B_r})$. As in Step 2, we conclude $u = \lim u^\varepsilon \in \mathcal{B}_1$, and $\Delta u = f$.

Step 4. In the general case, we will apply the *method of continuity* (see [2], Theorem 5.2). For $0 \le t \le 1$, we set $L_t = \Delta + t \cdot (L - \Delta)$, so that $L_0 = \Delta$, $L_1 = L$. Assumptions 3.1 and 5.1 for L remain valid for L_t as well with the same constants ν, K, and K_1. Therefore, the inequalities (6.2) are also true for $f = L_t u$, $0 \le t \le 1$.

From Step 3 we know that $L_0(\mathcal{B}_1) = \mathcal{B}_2$. Suppose that $L_s(\mathcal{B}_1) = \mathcal{B}_2$ for some $s \in [0,1]$. The inequalities (6.2) for $f = L_s u$ guarantee that L_s has the inverse mapping $L_s^{-1} : \ \mathcal{B}_2 \longrightarrow \mathcal{B}_1$. For $t \in [0,1]$ and $f \in \mathcal{B}_2$, the equation $L_t u = f$ is equivalent to the equation

$$L_s u = f + (t - s) \cdot (L - \Delta)u,$$

which in turn, is equivalent to the following one:

$$u = Tu = L_s^{-1} f + (t - s) \cdot L_s^{-1}(L - \Delta)u.$$

From (6.2) we obtain

$$\|T(u - v)\|_1 = (t - s) \cdot \|L_s^{-1}(L - \Delta)(u - v)\|_1$$

$$\le N_2|t - s| \cdot \|(L - \Delta)(u - v)\|_2 \le 2N_1N_2|t - s| \cdot \|u - v\|_1$$

for $u, v \in \mathcal{B}_1$. If $|t - s| < \delta = (2N_1N_2)^{-1}$, then $T : \mathcal{B}_1 \longrightarrow \mathcal{B}_1$ is a contraction mapping, hence there exist $u \in \mathcal{B}_1$ satisfying the equality $u = Tu$ which is equivalent to $L_t u = f$.

Since $f \in \mathcal{B}_2$ can be chosen arbitrarily, we have $L_t(\mathcal{B}_1) = \mathcal{B}_2$ provided $|t - s| < \delta$. By dividing the interval $[0,1]$ into subintervals of length less then δ, we conclude that $L_t(\mathcal{B}_1) = \mathcal{B}_2$ for all $t \in [0,1]$. In particular, for $t = 1$ we have $L(\mathcal{B}_1) = \mathcal{B}_2$. **QED**

Remark 6.1. If $\varphi \neq 0$, then we cannot guarantee the solvability of the problem (6.1) in $C^2(B_r) \cap C(\overline{B_r})$ under Assumptions 3.1 and 5.1. Indeed, all these assumptions are valid for the one-dimensional problem

$$u'' - x^{-2}\,u = 0 \ \text{ in } \ (0,1), \quad u(0) = u(1) = 1, \tag{6.6}$$

where we treat $(0,1)$ as $B_r = B_r(x_0)$ with $r = x_0 = 1/2$. However, for $u \in C^2(B_r) \cap C(\overline{B_r})$ from (6.6) it follows

$$u'' \asymp x^{-2}, \ \ u' \asymp -x^{-1}, \ \ u \asymp \log\frac{1}{x} \quad \text{as } x \to 0+,$$

contradicting $u(0) = 1$.

Theorem 6.2. *Let coefficients of linear elliptic operator* $L = \sum a_{ij}D_{ij} + \sum b_i D_i + c$ *satisfy Assumptions 3.1 in a ball* $B_r = B_r(x_0) \subset R^n$*, with some constants* $K \geq 0$, $0 < \nu \leq 1$, *and* $0 < \alpha < 1$. *Moreover, let* $a_{ij}, b_i, c, f \in C^\alpha(\overline{B_r})$, *and* $\varphi \in C(\overline{B_r})$. *Then the Dirichlet problem (6.1) has a unique solution* $u \in C^{2,\alpha;0}(B_r) \cap C(\overline{B_r})$.

Proof: First we consider the case $\varphi \in C^3(\overline{B_r})$. Setting $u = v + \varphi$, we reduce (6.1) to the equivalent problem

$$Lv = f_0 \ \text{ in } \ B_r, \quad v = 0 \ \text{ on } \ \partial B_r \tag{6.7}$$

with $f_0 = f - L\varphi \in C^\alpha(\overline{B_r}) \subset \mathcal{B}_2$. The previous theorem provides the solvability of the problems (6.1) and (6.7) in $\mathcal{B}_1 \subset C^{2,\alpha;0}(B_r) \cap C(\overline{B_r})$.

In general case $\varphi \in C(\overline{B_r})$, we approximate φ by polynomials $\{\varphi_m\}$ so that $|\varphi - \varphi_m| \leq 1/m$ on $\overline{B_r}$, $m = 1, 2, \cdots$. As in the proof of Theorem 5.2, the solutions of the problems

$$Lu_m = f \ \text{ in } \ B_r, \quad u_m = \varphi_m \ \text{ on } \ \partial B_r$$

compose a sequence $\{u_m\}$ convergent in $C(\overline{B_r})$ to a function $u \in C(\overline{B_r})$. Furthermore, applying Theorem 5.2 with $\gamma = 0$, we get $\|u_m\|^{(0)}_{2,\alpha} \leq A = const < \infty$ for all m, and then from Lemma 4.4 it follows $u \in C^{2,\alpha;0}(B_r)$, and $Lu = f$ in B_r. **QED**

7. The Dirichlet problem in a bounded domain

In this section, $\Omega \subset R^n$ is a bounded domain satisfying Assumptions 3.2 (the exterior sphere condition) with a constant $R > 0$, and also constants $\alpha, \beta \in (0,1)$ are fixed. We will extend the results of the previous section to the Dirichlet problem

$$Lu = f \ \text{ in } \ \Omega, \quad u = \varphi \ \text{ on } \ \partial\Omega. \tag{7.1}$$

The formulation of the next theorem is similar to Theorem 6.1 for $\Omega = B_r$.

Theorem 7.1. *Under Assumptions 3.1 and 5.1 on the coefficients of the linear elliptic operator* $L = \sum a_{ij}D_{ij} + \sum b_iD_i + c$*, for each* $f \in \mathcal{B}_2 = C^{0,\alpha;2-\beta}(\Omega)$ *there exists a unique* $u \in \mathcal{B}_1 = C^{2,\alpha;-\beta}(\Omega)$ *satisfying* $Lu = f$ *in* Ω*. In other words, L maps* $\mathcal{B}_1$ *onto* $\mathcal{B}_2$ *:* $L(\mathcal{B}_1) = \mathcal{B}_2$.

Proof: We fix $f \in \mathcal{B}_2 = C^{0,\alpha;2-\beta}(\Omega)$, and set $A = \|f\|^{(2-\beta)}_{0,\alpha;\Omega}$. For construction of a solution $u \in \mathcal{B}_1$ of $Lu = f$ in Ω, we will use a variant of the *Perron method of subsolutions.*

Step 1. For $x \in \Omega$ we have

$$|f(x)| \le (d_x/2)^{\beta-2}[f]^{(2-\beta)}_{0,0} \le Ad_x^{\beta-2}, \quad \text{where } d_x = \text{dist}(x, \partial\Omega). \tag{7.2}$$

By Theorem 3.5 there exists a subsolution $w \in C(\overline{\Omega})$ of $Lu = d_x^{\beta-2}$ in Ω, satisfying $0 \ge w \ge -N_1 d_x^\beta$ in Ω. Using the estimate (7.2), we see that the functions $U^0 = Aw$ and $-U^0$ are correspondingly subsolution and supersolution of $Lu = f$ in Ω, and

$$0 \ge U^0(x) \ge -N_1 A d_x^\beta, \quad x \in \Omega \tag{7.3}$$

with $N_1 = N_1(\nu, k, \beta, R, \text{diam}\Omega)$.

Step 2. Starting from U^0, we will introduce a sequence of subsolutions U^k as follows. First we fix a sequence $\{y^1, y^2, \cdots\} \subset \Omega$ which is dense in Ω. Then we set

$$\{x^1, x^2, x^3, x^4, x^5, x^6, \cdots\} = \{y^1, y^1, y^2, y^1, y^2, y^3, \cdots\},$$

so that each y^j appears infinitely many times in the sequence $\{x^k\}$, and we denote $d_k = \frac{1}{2}\text{dist}(x^k, \partial\Omega)$, $B^k = B(x^k) = B_{d_k}(x^k)$ for $k \ge 1$.

By virtue of Theorem 6.2, there exists a solution $u^1 \in C^{2,\alpha;0}(B^1) \cap C(\overline{B^1})$ of the problem

$$Lu^1 = f \text{ in } B^1, \quad u^1 = U^0 \text{ on } \partial B^1.$$

Since U^0 is a subsolution of $Lu = f$ in $\Omega \supset B^1$, we have $u^1 \ge U^0$ in B^1. Introducing the function U^1 by the equalities

$$U^1 = u^1 \text{ in } B^1, \quad U^1 = U^0 \text{ on } \overline{\Omega} \setminus B^1,$$

we have $U^1 \in C(\overline{\Omega})$, $U^1 \ge U^0$ on $\overline{\Omega}$. Moreover, Corollary 3.2 guarantees that U^1 is a subsolution of $Lu = f$ in Ω.

Repeating such a construction for $k = 2, 3, \cdots$, we introduce solutions u^k of the problems

$$Lu^k = f \text{ in } B^k, \quad u^k = U^{k-1} \text{ on } \partial B^k,$$

and function U^k by equalities

$$U^k = u^k \text{ in } B^k, \quad U^k = U^{k-1} \text{ on } \overline{\Omega} \setminus B^k.$$

So we obtain a sequence

$$U^0 \le U^1 \le U^2 \le \cdots \le U^k \le \cdots \tag{7.4}$$

of subsolutions of $Lu = f$ in Ω. Since $-U_0$ is a supersolution of $Lu = f$ in Ω, we also have

$$u^k \le -U^0 \text{ in } B^k, \quad U^k \le -U^0 \text{ in } \Omega \tag{7.5}$$

for all $k = 1, 2, \cdots$. Therefore, there exists

$$u(x) = \lim_{k\to\infty} U^k(x), \quad x \in \overline{\Omega}. \tag{7.6}$$

Step 3. We will show that $u \in \mathcal{B}_1$ and $Lu = f$ in Ω. First of all, from (7.3)–(7.6) it follows

$$|u(x)| \le \sup_k |U^k(x)| \le N_1 A d_x^\beta, \quad x \in \Omega. \tag{7.7}$$

Next let us fix $j \ge 1$, $d = d(y^j) = \frac{1}{2}\text{dist}(y^j, \partial\Omega)$, $B = B_d(y^j)$, and choose a subsequence $\{x^{k_i}\}$ of $\{x^k\}$ such that $x^{k_i} = y^j$ for all $i = 1, 2, \cdots$. Then $B^{k_i} = B$, and functions $u^{k_i} \in C^{2,\alpha;0}(B) \cap C(\overline{B})$ satisfy $Lu^{k_i} = f$ in B for all $i = 1, 2, \cdots$. Further,

$$[f]^{(2)}_{0,\alpha;B} \le d^{2+\alpha}[f]_{\alpha;B} \le d^\beta [f]^{(2-\beta)}_{0,\alpha;\Omega} \le A d^\beta.$$

Therefore, from Theorem 5.2 with $\gamma = 0$ and (7.7) we have

$$\|u^{k_i}\|^{(0)}_{2,\alpha;B} \le N \cdot \left(\sup_B |u^{k_i}| + [f]^{(2)}_{0,\alpha;B} \right) \le N A d^\beta, \ \ i = 1, 2, \cdots.$$

Applying Lemma 4.4, we obtain $u = \lim u^{k_i} \in C^{2,\alpha;0}(B)$, $Lu = f$ in B, and

$$\|u\|^{(0)}_{2,\alpha;B} \le N A d^\beta. \tag{7.8}$$

Since $\cup_j B(y^j) = \Omega$, we have $u \in C^{2,\alpha}_{loc}(\Omega)$, and $Lu = f$ in Ω.

Step 4. In order to show that $u \in \mathcal{B}_1 = C^{2,\alpha;-\beta}(\Omega)$ let us notice that (7.8) yields

$$\sum_{k=0}^{2} d^k [u]_{k,0;B_{d/2}} + d^{2+\alpha}[u]_{2,\alpha;B_{d/2}} \le N A d^\beta \tag{7.9}$$

for concentric ball $B_{d/2} = B_{d/2}(y^j)$ of radius $d/2 = \frac{1}{4}\text{dist}(y^j, \partial\Omega)$.

We will first evaluate u in the norm of $C^{2,0;-\beta}(\Omega)$. By virtue of (1.15) and (1.16), for each $k = 0, 1, 2$, there exist $x_0 \in \Omega$, $|l| = k$, and $x \in B(x_0)$, such that

$$\frac{1}{2}[u]^{(-\beta)}_{k,0;\Omega} \le d^{k-\beta}(x_0) \cdot |D^l u(x)|. \tag{7.10}$$

Since $\{y^j\}$ is dense in Ω, we can choose y^j close to x, so that $x \in B_{d/2}(y^j)$, $d(x_0) \le 2d$, where $d = d(y^j)$. Then (7.9) and (7.10) give us

$$[u]^{(-\beta)}_{k,0;\Omega} \le 2^{k+1} d^{k-\beta} [u]_{k,0;B_{d/2}} \le NA, \quad k = 0, 1, 2. \tag{7.11}$$

We have proved that $u \in C^{2,0;-\beta}(\Omega)$. Analogously, there exist $x_0 \in \Omega$, $|l| = 2$, and $x, y \in B(x_0)$ satisfying

$$\frac{1}{2}[u]^{(-\beta)}_{2,\alpha;\Omega} \le d^{2+\alpha-\beta}(x_0) \frac{|D^l u(x) - D^l u(y)|}{|x-y|^\alpha}. \tag{7.12}$$

If $|x - y| < \frac{1}{4} d(x_0)$, then we can choose y^j such that $x, y \in B_{d/2}(y^j)$, $d(x_0) \le 2d$, so that (7.12) together with (7.9) provide us the estimate

$$[u]^{(-\beta)}_{2,\alpha;\Omega} \le 2^{3+\alpha} d^{2+\alpha-\beta} [u]_{2,\alpha;B_{d/2}} \le NA.$$

Finally, if $|x - y| \ge \frac{1}{4} d(x_0)$, then from (7.12) we get

$$[u]^{(-\beta)}_{2,\alpha;\Omega} \le 2 \cdot 4^\alpha d^{2-\beta}(x_0) |D^l u(x) - D^l u(y)|$$

$$\le 4^{1+\alpha} d^{2-\beta}(x_0) [u]_{2,0;B(x_0)} \le 4^{1+\alpha} [u]^{(-\beta)}_{2,0;\Omega} \le NA,$$

that completes the proof. **QED**

Theorem 7.2. *Let coefficients of linear elliptic operator* $L = \sum a_{ij} D_{ij} + \sum b_i D_i + c$ *satisfy Assumptions 3.1 in bounded domain* $\Omega \subset R^n$, *satisfying Assumptions 3.2. Moreover, let* $a_{ij}, b_i, c, f \in C^\alpha(\overline{\Omega})$, *and* $\varphi \in C(\overline{\Omega})$. *Then the Dirichlet problem (6.1) has a unique solution* $u \in C^{2,\alpha;0}(\Omega) \cap C(\overline{\Omega})$.

Proof is quite the same as the proof of Theorem 6.2, with Ω in place of B_r.

Remark 7.1. The technique exposed in this paper is also applicable to investigation of $C^{2,\alpha}$-smoothness of solutions near the boundary of a smooth domain, assuming the appropriate smoothness of the boundary values of solutions. For example, if in addition to conditions of Theorem 7.2, we assume that $\varphi \in C^{2,\alpha}(\overline{\Omega})$, and $\partial\Omega \in C^{2,\alpha}$ (i.e. $\partial\Omega$ is locally represented by means of functions of class $C^{2,\alpha}$), then the solution of the problem (7.1) belongs to $C^{2,\alpha}(\overline{\Omega})$. After subtraction of φ and local "flattening" of $\partial\Omega$ by a transformation of class $C^{2,\alpha}$, the boundary $C^{2,\alpha}$-estimates are reduced to the estimates near the flat portion Γ of the boundary $\partial\Omega$, provided $u = 0$ on Γ. Such estimates can be obtained by means of weighted Hölder spaces with $d(x) = \frac{1}{2}\mathrm{dist}(x, \partial\Omega \setminus \Gamma$.

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