A bifurcation approach to non-planar traveling waves in reaction-diffusion systems

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Contents

1 Introduction

A striking phenomenon in the dynamics of nonlinear, spatially extended systems is the propagation of coherent interfaces that monitor the competition between metastable states. A prototypical example of such systems are reaction-diffusion systems, describing for instance the effect of diffusion, reaction, and convection on combustion fronts [16], or arising as phenomenological models in areas ranging from phase boundary motion in material science [3, 4]

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to the creation of high current filaments in semiconductors [15]. Many of these models can be written in the general form

$$u_t = \nabla \cdot (a\nabla u) + f(u, \nabla u) + c(n \cdot \nabla)u, \tag{1}$$

in which u is the vector of chemical concentrations, $u = (u^m)_{1\leq m\leq N} \in \mathbb{R}^N$, and depends upon time t and space $x = (x_1, x_2) \in \mathbb{R}^2$. The diffusion matrices should be elliptic, $a = (a^m_{ij})^{1\leq m\leq N}_{1\leq i,j\leq 2}$ such that

$$(\nabla \cdot (a\nabla u))^m = a^m_{ij}\partial_{ij}u^m, \quad \sum_{ij} a^m_{ij} y_i y_j \geq M \sum_i y_i^2,$$

for some positive constant $M > 0$ independent of $y = (y_1, y_2)$. For our purpose, we will work with smooth nonlinearities f, without further assumptions on the specific shape. The last term in (1) is induced by passing to a moving coordinate frame in the direction $n = (\sin\varphi, \cos\varphi) \in S^1$ with speed c, so that steady solutions to (1) are traveling waves propagating in the direction n with speed c. This last term could be absorbed into the nonlinearity f but we prefer to preserve the idea of a fixed (laboratory) reference frame for our considerations, in particularly when varying the speed c. We will sometimes restrict to the *isotropic case*, when $a^m_{ij} = d^m > 0$ and $f = f(u)$.

The propagation of one-dimensional (planar) waves in such systems has been studied intensively and has been well understood in many different settings. Such planar waves $u = u(n\cdot x)$ solve an ordinary differential equation, for which a variety of tools are available to study the existence of bounded orbits, such as singular perturbation theory, Conley index theory, or bifurcation methods. Typical planar waves are traveling-wave solutions connecting two homogeneous equilibria.

Our interest here is directed to traveling waves that are *not planar*, $u = u(n \cdot x, n^\perp \cdot x)$. These solutions cannot be found as solutions to an ODE since they depend on more than one spatial variable. However, we will use a well adapted reduction method to find *near planar* traveling waves as solutions to an ordinary differential equation. Whereas the primary, planar traveling wave can be found as a solution to the ODE in the variable $n \cdot x$, the existence and shape of the *almost planar* traveling wave will be determined by a reduced ODE in the transverse variable $n^\perp \cdot x$.

The starting point of our approach to non-planar waves is the following hypothesis asserting existence of a planar wave connecting two homogenenous equilibria.

Hypothesis 1.1 (Existence of a planar wave) *We assume that there exist a direction $n = n_*$, a positive speed $c = c_* \geq 0$, and asymptotic states $q_\pm$ such that (1) possesses a smooth planar traveling-wave solution $u(t, x) = q_*(n_* \cdot x)$ connecting q_- and q_+, i.e.*

$$q_*(\zeta) \to q_\pm, \quad \text{as} \quad \zeta \to \pm\infty.$$

The profile q_* solves

$$(an \cdot n)u'' + f(u, nu') + cu' = 0, \tag{2}$$

with $n = n_*$, $c = c_*$, where $'$ denotes differentiation with respect to $\zeta = n \cdot x$, the vector $(an \cdot n)$ is regarded as a diagonal $N \times N$-matrix, and nu' stands for the tensor product.

Now given a planar wave q_*, we characterize almost planar interfaces in the following way.

Definition 1.2 (Almost planar interfaces) Set $\xi := n^\perp \cdot x$ the direction perpendicular to the direction of propagation, in which $n^\perp = n'(\varphi) = (\cos\varphi, -\sin\varphi)$. We call a solution u to (1) an *almost planar interface δ-close to q_**, for some $\delta > 0$, if u is of the form

$$u(x) = q_*(\zeta + h(\xi)) + u_1(\xi, \zeta), \tag{3}$$

with $h \in C^2(\mathbb{R})$ and

$$\sup_{\xi\in\mathbb{R}} |h'(\xi)| < \delta, \quad \sup_{\xi\in\mathbb{R}} \|u_1(\xi,\cdot)\|_{H^1(\mathbb{R},\mathbb{R}^N)} < \delta, \quad |c - c_*| < \delta.$$

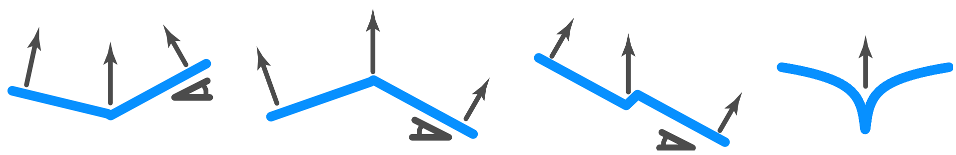
We say that u is a *planar interface* if $h'' \equiv 0$.

Among these almost planar interfaces we distinguish the class of asymptotically planar interfaces, to which we refer as corner defects.

Definition 1.3 (Corner defects) We call u a *corner defect* if it is of the form (3), with $h'' \not\equiv 0$, and $h'(\xi) \to \eta_\pm$ as $\xi \to \pm\infty$. We say that

- u is an *interior corner* if $\eta_+ < \eta_-$;
- u is an *exterior corner* if $\eta_- < \eta_+$;
- u is a *step* if $\eta_+ = \eta_- \neq 0$;
- u is a *hole* if $\eta_+ = \eta_- = 0$;

see Figure 1.



(a) (b) (c) (d)

Fig. 1 Schematic plot of the four different types of corner defects: interior corner (a), exterior corner (b), step (c), and hole (d). The middle arrows indicate the speed of the defect, whereas the left and right arrows indicate the normal speed of propagation of the asymptotic planar interfaces. The asymptotic slopes are $\eta_\pm = \tan\vartheta_\pm$.

In this paper we review a number of scenarios discussed in [6, 7], which show that these four types of corner defects can be found in systems of the form (1) under suitable assumptions on the primary, planar wave q_*. All these defects share the property that the *angle* of the interface at each point relative to the primary interface is small. They are constructed with the help of methods from bifurcation theory which do not rely upon monotonicity arguments or comparison principles. We mention that comparison principles have been used to show existence of interior corners for large angles in scalar equations [2, 5, 11].

It is worth mentioning that similar results hold for planar traveling wave trains as well, more precisely, when the planar traveling wave q_* in Hypothesis 1.1 is a periodic function, $q_*(\zeta) = q_*(\zeta + L)$ for all ζ and some $L > 0$. We may then mimic the analysis in this article, replacing the spaces of functions defined on the whole real line by spaces of L-periodic functions.

2 Planar waves

The construction of almost planar interfaces relies upon a number of properties of the planar wave that we describe in this section.

Linear dispersion

The first properties concern the *linear stability* of the planar wave q_* with respect to first one-dimensional, longitudinal, and then also two-dimensional, transverse perturbations. We linearize the system (1) at the solution q_* and obtain the linearized operator

$$\mathcal{L}_* u = \nabla \cdot (a\nabla u) + f_1^* u + f_2^* \nabla u + c_*(n_* \cdot \nabla)u,$$

where

$$f_1^* = \partial_1 f(q_*, n_* q_*'), \quad f_2^* = \partial_2 f(q_*, n_* q_*').$$

With the Fourier decomposition into transverse wavenumbers k, $u = v(n_* \cdot x)\mathrm{e}^{\mathrm{i}k(n_*^\perp \cdot x)}$, we find the family of linear operators

$$\mathcal{L}(k)v = (an_* \cdot n_*)v'' + \mathrm{i}k((a+a^T)n_* \cdot n_*^\perp)v' - k^2(an_*^\perp \cdot n_*^\perp)v + f_1^* v + f_2^*(n_* v' + \mathrm{i}kn_*^\perp v) + c_* v'.$$

At $k = 0$, we recover the one-dimensional linearized problem

$$\mathcal{L}_0 v := (an_* \cdot n_*)v'' + f_1^* v + f_2^*(n_* v') + c_* v'. \tag{4}$$

Hypothesis 2.1 (Stability in one dimension) *We assume that*

(i) the spectrum of $\mathcal{L}_0$ is contained in $\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda < 0\} \cup \{0\}$;

(ii) $\mathcal{L}_0$ is Fredholm with index zero and has a one-dimensional generalized kernel spanned by q_*'.

Furthermore, we assume that for some positive constant k_* the spectrum of $\mathcal{L}(k)$, $k \neq 0$, satisfies

(iii) it is contained in $\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda < 0\} \cup \{\lambda(k)\}$, for $|k| \leq k_*$, in which $\lambda(k) \in \mathbb{C}$ is the continuation of the algebraically simple eigenvalue $\lambda = 0$ of $\mathcal{L}(0)$, for small k;

(iv) it is contained in $\{\lambda \in \mathbb{C} : \ \mathrm{Re}\,\lambda < 0\}$, for $|k| > k_*$.

The first two assumptions in the above hypothesis imply stability of the planar wave with respect to perturbations that depend only on $n_* \cdot x$, while the latter ones concern stability with respect to transverse perturbations. We will later sharpen assumption (iii) and find that the type of corner defects critically depends on the precise shape of the *linear dispersion relation* $\lambda(k)$, and the *(linear) group velocity* $c_{\mathrm{g}} := \mathrm{i}\lambda'(0)$. We note in passing that the symmetry $\lambda \mapsto \bar{\lambda}$, $k \mapsto -k$, implies that $\mathrm{Re}\,\lambda$ and $\mathrm{Im}\,\lambda$ are even and odd in k, respectively. In particular, the even, respectively odd, order derivatives of λ at $k = 0$ are real, respectively purely imaginary, so that the group velocity is real and stability is to leading order determined by the sign of $\lambda''(0)$. In the case of isotropic media the linear dispersion has the additional symmetry $k \mapsto -k$, and then λ is a real-valued even function of k. In particular, the group velocity always vanishes in this case.

Nonlinear and directional dispersion

We now focus on the one-dimensional problem (2), which we consider with parameters n and c. It turns out that the stability Hypothesis 2.1 implies existence of planar waves for nearby directions of propagation $n \sim n_*$, more precisely, we have the following result.

Lemma 2.2 (Planar waves [7, Lemma 2.4]) *Assume that Hypotheses 1.1 and 2.1 hold. Then there exists a positive constant ε, such that for each angle φ with $|\varphi - \varphi_*| < \varepsilon$, the one-dimensional system (2) possesses a traveling-wave solution $q(\varphi)$ that connects q_- and q_+, and which propagates with speed $c = c(\varphi)$ in the direction $n = n(\varphi)$. Moreover, each of these traveling-waves satisfies Hypothesis 2.1 on stability.*

This result is obvious in the case of isotropic media where, due to the rotation invariance, we have $c(\varphi) = c_*$ and $q(\varphi) = q_*$.

We call $c = c(\varphi)$ the *nonlinear dispersion relation*. A short computation shows that linear and nonlinear dispersion coincide at first order, $c'(\varphi_*) = c_{\mathrm{g}}$ ([7, Section 2]).

The nonlinear dispersion $c = c(\varphi)$ gives the speed of propagation of a planar wave in the direction $n(\varphi)$. In addition, we consider the speed of propagation in the direction $n = n_*$ of the primary wave q_*,

$$d(\varphi) = \frac{c(\varphi)}{\cos(\varphi - \varphi_*)}.$$

We call $d = d(\varphi)$ *directional dispersion* or *flux*. Moreover, we say that

- d is *convex* if $d'' > 0$;
- d is *concave* if $d'' < 0$;
- d is *flat* if $d'' \equiv 0$;

for angles φ in a neighborhood of φ_*. Notice that in the particular case of isotropic media the flux is always convex, $d(\varphi) = c_*/\cos(\varphi - \varphi_*)$.

Together with the assumptions on the linear dispersion $\lambda(k)$, convexity properties of the flux d are essential in the construction of almost planar waves below. In order to see this, recall that, in particular, interior and exterior corners are by definition *asymptotically planar*. Assuming the existence of such a corner defect propagating with speed c in the direction n_* of the planar wave, at $\xi = \pm\infty$ we find planar interfaces $q_\pm(\zeta + \eta_\pm \xi)$. The normal directions to these interfaces $n_\pm = (\sin\varphi_\pm, \cos\varphi_\pm)$ are obtained from the equality $\tan(\varphi_\pm - \varphi_*) = \eta_\pm$, and their normal propagation speed from the nonlinear dispersion relation $c_\pm = c(\varphi_\pm)$. But since both interfaces propagate with speed c in the direction n_* of the primary interface, the angles $\varphi_\pm$ satisfy,

$$c = d(\varphi_\pm).$$

In particular, for an interior/exterior corner this equation must have two distinct solutions $\varphi_\pm$ which in the case of small angles $\varphi_\pm \sim \varphi_*$ requires a non-flatness condition on the flux d, for angles φ close to φ_*. In other words, a flat flux d appears to exclude existence of interior and exterior corners.

Nonlinear dispersion curve

A convenient way to represent wave propagation in anisotropic media is to plot the nonlinear dispersion relation $c = c(\varphi)$ as a curve in the plane. We can plot the vector $c(\varphi)n(\varphi)$ and let φ vary, which then traces a curve Γ parameterized through

$$\gamma(\varphi) = c(\varphi)(\sin\varphi, \cos\varphi) = d(\varphi)\cos(\varphi - \varphi_*)(\sin\varphi, \cos\varphi).$$

We call this curve the *nonlinear dispersion curve*. We will see below how properties of dispersion relations and fluxes can be read off this curve, so that we can predict existence of corners from geometric properties of this curve ([7, Section 7]).

First, a vanishing group velocity $c_{\rm g}$ is equivalent to a first order tangency between Γ and the circle with diameter the vector $c_* n_*$, or, more generally, any circle with diameter along n_* passing through $\gamma(\varphi_*)$. Next, consider the circle C_* with diameter joining the points $\gamma(\varphi_*) = c_* n_*$ and $c_{\rm g} n_*^\perp$. Then we have that the flux

- *d is convex if Γ lies outside C_*;*
- *d is concave if Γ lies inside C_*;*
- *d flat if Γ coincides with C_*;*

for φ in a neighborhood of φ_*; see Figure 2. Next, fix a speed c and the circle C with diameter

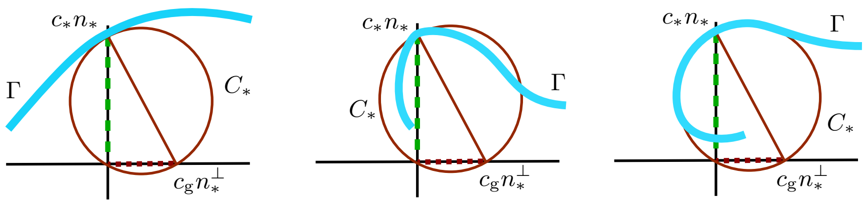


Fig. 2 Plot of the nonlinear dispersion curve Γ and of the circle C_* in cases of convex, concave, and flat flux d.

given by the segment connecting cn_* with the origin. Then any intersection point of the circle C with the curve Γ corresponds to a planar interface moving with speed c in the direction n_*, since the speed of such an interface in this direction is given by $c(\varphi)/\cos(\varphi - \varphi_*)$. Consequently, the asymptotic planar interfaces to a corner defect propagating in a direction n_* with speed c, are necessarily among the intersection points of the curve Γ with the circle C. Furthermore, in the simplest case of a stable planar wave q_*, when $\mathrm{Re}\,\lambda(k) < 0$, for $k \neq 0$, it turns out that

- *if the corner defect is an interior corner, then the curve Γ lies inside the circle C for angles φ between the asymptotic angles $\varphi_+ < \varphi_-$;*
- *if the corner defect is an exterior corner, then the curve Γ lies outside the circle C for angles φ between the asymptotic angles $\varphi_- < \varphi_+$;*

see Figure 3. We emphasize that these geometric properties remain valid for large angles φ, as well.

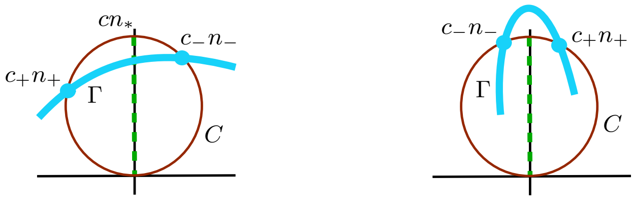


Fig. 3 Plot of the nonlinear dispersion curve Γ and of the circle C in the cases of interior (left) and exterior corners (right). The intersection points $c_\pm n_\pm$ represent the asymptotic planar interfaces of the defect with $\varphi_+ < \varphi_* < \varphi_-$ for interior corners (left), and $\varphi_- < \varphi_* < \varphi_+$ for exterior corners (right). (The curve Γ is oriented clockwise.)

3 Spatial dynamics and reduction

The search of almost planar interfaces relies upon a spatial dynamics formulation with an appropriate parameterization for relative equilibria and a center manifold reduction.

We fix the direction $n = n_*$ and write the stationary equation to (1) with $n = n_*$ as a dynamical system in which the time-like variable is $\xi = n_*^\perp \cdot x$, the direction perpendicular to the direction of propagation of the planar wave. In the coordinates $(\xi, \zeta) = (n_*^\perp \cdot x, n_* \cdot x)$ we find the first-order system

$$\mathbf{u}_\xi = \mathcal{A}(c)\mathbf{u} + \mathcal{F}(\mathbf{u}), \tag{5}$$

in which $\mathbf{u} = (u, v)^T$, with $v = u_\xi$, and

$$\mathcal{A}(c) = \begin{pmatrix} 0 & \mathrm{id} \\ -(an_*^\perp \cdot n_*^\perp)^{-1}\left((an_* \cdot n_*)\partial_{\zeta\zeta} + c\partial_\zeta\right) & -(an_*^\perp \cdot n_*^\perp)^{-1}((a + a^T)n_* \cdot n_*^\perp)\partial_\zeta \end{pmatrix},$$

$$\mathcal{F}(\mathbf{u}) = \begin{pmatrix} 0 \\ -(an_*^\perp \cdot n_*^\perp)^{-1} f(u, n_* u_\zeta + n_*^\perp v) \end{pmatrix}.$$

The Hypothesis 1.1 on existence together with the translation invariance of the system (1) imply that at $c = c_*$ the dynamical system (5) has a line of equilibria obtained from the primary planar wave $u = q_*(\cdot)$ together with the translations $q_*(\cdot + h)$,

$$\mathbf{q}_*^h = \begin{pmatrix} q_*(\cdot + h) \\ 0 \end{pmatrix}. \tag{6}$$

The almost planar waves are obtained as bounded orbits to the infinite-dimensional dynamical system (5) bifurcating from this line of equilibria for c close to c_*. We therefore consider the linearization of (5) about $\mathbf{q}_*^0$ at $c = c_*$,

$$\mathcal{A}_* = \begin{pmatrix} 0 & \mathrm{id} \\ 2\mathcal{L}''(0)^{-1}\mathcal{L}_0 & -2\mathcal{L}''(0)^{-1}(\mathrm{i}\mathcal{L}'(0)) \end{pmatrix},$$

in which $\mathcal{L}_0$ is the linear operator defined in (4), $\mathcal{L}'(0)$ represents the derivative with respect to k of $\mathcal{L}(k)$ at $k = 0$,

$$\mathrm{i}\mathcal{L}'(0)v = -((a + a^T)n_* \cdot n_*^\perp)v' - f_2^*(n_*^\perp v),$$

and $\mathcal{L}''(0)$ the second derivative given by

$$\mathcal{L}''(0) = -2(an_*^\perp \cdot n_*^\perp).$$

We consider $\mathcal{A}_*$ as a closed linear operator in $Y = (H^1 \times L^2)(\mathbb{R}, \mathbb{R}^N)$ with domain of definition $Y^1 = (H^2 \times H^1)(\mathbb{R}, \mathbb{R}^N)$.

The spectral properties of $\mathcal{A}_*$ are closely related to those of the operators $\mathcal{L}(k)$ in Hypothesis 2.1 (e.g., see [6, 7, 14]). Under Hypothesis 2.1, the spectrum of $\mathcal{A}_*$ splits into

$$\mathrm{spec}(\mathcal{A}_*) = \mathrm{spec}_0(\mathcal{A}_*) \cup \mathrm{spec}_1(\mathcal{A}_*), \tag{7}$$

with $\mathrm{spec}_1(\mathcal{A}_*)$ bounded away from the imaginary axis,

$$\mathrm{spec}_1(\mathcal{A}_*) \subset \{\nu \in \mathbb{C} : \ |\,\mathrm{Re}\,\nu| \geq \epsilon\},$$

for some positive constant ϵ, and $\mathrm{spec}_0(\mathcal{A}_*)$ consisting of a finite number of purely imaginary eigenvalues with finite algebraic multiplicity. These eigenvalues are precisely given by the zeros of the linear dispersion $\lambda(k)$,

$$\mathrm{spec}_0(\mathcal{A}_*) = \{\pm \mathrm{i}k, \ \lambda(k) = 0\},$$

and the algebraic multiplicity of an eigenvalue $\mathrm{i}k$ is equal to the multiplicity of the root k of $\lambda(k) = 0$. Notice that nonzero eigenvalues occur in pairs, since the operator is real, and that 0 is always an eigenvalue, since $\lambda(0) = 0$.

The above splitting of the spectrum suggests that the dynamical system (5) possesses a finite-dimensional center manifold containing the almost planar waves. For the construction of the center manifold we proceed in the following way. Assume that the number of the purely imaginary eigenvalues counted with multiplicities of $\mathcal{A}_*$ is $m+1$, with $m \geq 0$ since zero always belongs to the spectrum. We choose $m+1$ linearly independent generalized eigenvectors of $\mathcal{A}_*$ corresponding to the purely imaginary eigenvalues, $\{\mathbf{e}_0, \ldots, \mathbf{e}_m\}$, with $\mathbf{e}_0 = (q_*', 0)$ the eigenvector in the kernel of $\mathcal{A}_*$. We use these vectors to construct the (unique) spectral projection P onto the linear subspace spanned by these vectors, and then proceed similarly for the shifted equilibria $\mathbf{q}_*^h$ by introducing the shifted linear operator $\mathcal{A}_*^h$, the shifted eigenvectors $\{\mathbf{e}_0^h, \ldots, \mathbf{e}_m^h\}$, and the shifted projection P^h.

Following the general strategy for reduction around a continuous family of equilibria [13], we decompose

$$\mathbf{u} = \mathbf{q}_*^h + \eta_1 \mathbf{e}_1^h + \cdots + \eta_m \mathbf{e}_m^h + \mathbf{w}^h, \text{ with } P^h \mathbf{w}^h = P\mathbf{w} = 0. \tag{8}$$

Here h and η_j, $j = 1, \ldots, m$, are real functions depending upon ξ. This provides us with coordinates in a neighborhood of the family of equilibria $\mathbf{q}_*^h$. Then substituting (8) into (5), we obtain a first-order system for h, η_j, and $\mathbf{w}$, in which the translation invariance in ζ is used to eliminate the shift $(\cdot)^h$ from each equation. Finally, we are left with an equation for h,

$$h_\xi = f_0(\eta_1, \ldots, \eta_m, \mathbf{w}; c), \tag{9}$$

which decouples, and a first-order system

$$\begin{aligned} \eta_{j\xi} &= f_j(\eta_1, \ldots, \eta_m, \mathbf{w}; c), \quad j = 1, \ldots, m, \\ \mathbf{w}_\xi &= f_w(\eta_1, \ldots, \eta_m, \mathbf{w}; c), \end{aligned} \tag{10}$$

in which the vector field does not depend upon h. At this point, we use a center manifold reduction theorem [10] to conclude that for c close to c_*, all solutions with η_j and $\mathbf{w}$ sufficiently small have

$$\mathbf{w} = \Psi(\eta_1, \ldots, \eta_m; c),$$

with Ψ a reduction function. Then these solutions can be found by solving the reduced system for η_j obtained by substituting $\mathbf{w} = \Psi(\eta_1, \ldots, \eta_m; c)$ into (10),

$$\eta_{j\xi} \quad = \quad \widetilde{f}_j(\eta_1, \ldots, \eta_m; c), \quad j = 1, \ldots, m. \tag{11}$$

The reduction theorem ensures that all nonlinear functions and their dependence upon parameters are of class C^k for an arbitrary but fixed $k < \infty$.

Bounded solutions to the reduced system (11) correspond to almost planar waves to the original system. More precisely, we have the following correspondence

equilibrium	$\longleftrightarrow$	planar wave
heteroclinic orbit	$\longleftrightarrow$	interior/exterior corner
homoclinic orbit to a nontrivial equilibrium	$\longleftrightarrow$	step
homoclinic orbit to zero	$\longleftrightarrow$	hole

In particular, the family of planar waves $q(\varphi)$ in Lemma 2.2 provides us with a family of equilibria to the reduced system for $c = c(\varphi)$. Bounded orbits to the reduced system, other than these equilibria, can be found by computing the vector field at lowest order. It turns out that the coefficients in the Taylor expansion of the vector field depend upon the linear dispersion $\lambda(k)$ and the flux $d(\varphi)$, which then determine the type of almost planar interfaces in (1). In the following sections, we will make precise assumptions on $\lambda(k)$ and $d(\varphi)$ that allow us to analyze the reduced system of ODEs.

4 Stable interior and exterior corners

In this section, we consider the simplest situation of a planar wave which is stable in two space-dimensions. Throughout this section we make the following hypothesis.

Hypothesis 4.1 (Stability in two dimensions) *Assume that Hypotheses 1.1 and 2.1 hold, and in addition that the linear dispersion satisfies*

$$\lambda''(0) < 0, \quad \operatorname{Re} \lambda(k) < 0, \quad \forall\, |k| \leq k_*,\ k \neq 0.$$

Clearly, this hypothesis implies linear stability of the planar wave in two space-dimensions.

We now follow the strategy of proof described in Section 3. The spectrum of $\mathcal{A}_*$ decomposes into

$$\operatorname{spec}(\mathcal{A}_*) = \{0\} \cup \operatorname{spec}_1(\mathcal{A}_*),$$

with $\operatorname{spec}_1(\mathcal{A}_*)$ bounded away from the imaginary axis as before, and 0 an isolated eigenvalue with geometric multiplicity one. The algebraic multiplicity of 0 is equal to the multiplicity of the root $k = 0$ of $\lambda(k)$. Since $\lambda''(0) < 0$ the multiplicity is at most two, and equal to two precisely when the group velocity vanishes, $c_{\mathrm{g}} = 0$.

Almost planar waves in case of non-zero group velocity

Assume that $c_{\mathrm{g}} \neq 0$, so that 0 is a simple eigenvalue of $\mathcal{A}_*$. Then the Ansatz (8) becomes

$$\mathbf{u} = \mathbf{q}_*^h + \mathbf{w}^h, \text{ with } P^h\mathbf{w}^h = P\mathbf{w} = 0,$$

and we find the system

$$\begin{aligned} h_\xi &= \mathrm{O}(|c - c_*| + |\mathbf{w}|_{Y^1}^2), \\ \mathbf{w}_\xi &= \mathcal{A}_*\mathbf{w} + \mathrm{O}(|c - c_*| + |\mathbf{w}|_{Y^1}^2). \end{aligned}$$

Here the linear operator $\mathcal{A}_*$ is hyperbolic on $\mathrm{id} - P$, so that the equation for $\mathbf{w}$ has no small bounded solutions, except for one equilibrium near $\mathbf{w} = 0$, for c close to c_*. Consequently, the only solutions for which the derivative h_ξ stays bounded are those with $h_\xi \equiv \mathrm{const.}$, and we conclude that there are no almost planar interfaces in this case.

However, the condition on vanishing group velocity c_{g} can be interpreted in a slightly different fashion. We can pass to a co-moving frame $x \mapsto x - c_{\mathrm{g}} n_*^\perp t$ in the system (1), which adds a drift term $c_{\mathrm{g}}(n_*^\perp \cdot \nabla)u$ to the nonlinearity f. With this new nonlinearity, a straightforward computation shows that $\widetilde{c}_{\mathrm{g}} = 0$. The nonlinear dispersion relation is easily obtained from the speed of propagation in the normal direction, corrected by $-c_{\mathrm{g}} \sin(\varphi - \varphi_*)$, such that

$$\widetilde{d}(\varphi) = \frac{c(\varphi) - c_{\mathrm{g}} \sin(\varphi - \varphi_*)}{\cos(\varphi - \varphi_*)} = d(\varphi) - c_{\mathrm{g}} \tan(\varphi - \varphi_*).$$

In particular, $\widetilde{d}''(\varphi_*) = d''(\varphi_*)$, so that this change of variables does not influence upon the convexity or flatness of the flux d. Consequently, the group velocity c_{g} always vanishes in a suitable frame of reference, and we therefore assume from now on that $c_{\mathrm{g}} = \mathrm{i}\lambda'(0) = 0$.

Existence of corner defects

Assume that $c_{\mathrm{g}} = 0$. Then the eigenvalue of $\mathcal{A}_*$ in the origin is algebraically double with kernel and generalized kernel spanned by

$$\ker \mathcal{A}_* = \mathrm{span}(\mathbf{e}_0),\ \mathbf{e}_0 = \begin{pmatrix} q_*' \\ 0 \end{pmatrix}, \quad \mathrm{gker}\, \mathcal{A}_* = \mathrm{span}(\mathbf{e}_0, \mathbf{e}_1),\ \mathcal{A}_*\mathbf{e}_1 = \mathbf{e}_0.$$

We decompose

$$\mathbf{u} = \mathbf{q}_*^h + \eta\mathbf{e}_1^h + \mathbf{w}^h, \text{ with } P^h\mathbf{w}^h = P\mathbf{w} = 0. \tag{12}$$

Employing center-manifold reduction as explained in Section 3, we obtain the reduced equation for η,

$$\eta_\xi = \frac{2}{\lambda''(0)} \left((c - c_*) - \frac{d''(\varphi_*)}{2}\eta^2 \right) + \mathrm{O}(|c - c_*|(|c - c_*| + |\eta|) + |\eta|^3). \tag{13}$$

This is a scalar first-order ODE, so that any bounded orbit is either an equilibrium or a heteroclinic orbit. Consequently, nontrivial almost planar interfaces are either interior or exterior corners. A straightforward analysis of this equation then leads to the following existence result.

Theorem 4.2 (Existence of interior/exterior corners [7, Theorem 1]) *We assume existence of a planar interface q_* propagating in the direction n_* with speed c_*, Hypothesis 1.1, that satisfies the stability assumptions, Hypotheses 2.1, 4.1, and that the group velocity $c_{\mathrm{g}} = 0$. Then there exists a positive constant $\delta > 0$ such that the following hold.*

- *If the flux d is convex, then for each speed $c > c_*$, $|c - c_*| < \delta$, there exists an interior corner defect propagating in the direction n_* which is unique up to translation in x. For $c < c_*$, $|c - c_*| < \delta$, there are no almost planar traveling waves.*
- *If the flux d is concave, then for each speed $c < c_*$, $|c - c_*| < \delta$, there exists an exterior corner defect propagating in the direction n_* which is unique up to translation in x. For $c > c_*$, $|c - c_*| < \delta$, there are no almost planar traveling waves.*
- *If the flux d is flat, there are no almost planar traveling waves propagating in the direction n_*, for any speed c with $|c - c_*| < \delta$.*

Furthermore, the corner defects above are to leading order given by

$$q(x) = q_*(\zeta + h(\xi)) + \mathrm{O}(|c - c_*|),$$

in which $\zeta = n_* \cdot x$, $\xi = n_*^\perp \cdot x$, and the derivative of h satisfies

$$h'(\xi) = \frac{\lambda''(0)}{d''(\varphi_*)}\,\beta \tanh(\beta\xi) + \mathrm{O}(|c - c_*|\mathrm{e}^{-2|\beta\xi|}), \quad \beta = \frac{\sqrt{2(c - c_*)d''(\varphi_*)}}{\lambda''(0)} < 0.$$

Stability

The corner defects found in Theorem 4.2 are asymptotically stable with respect to perturbations which are exponentially localized in the direction perpendicular to the direction of propagation. We refer to [6, Theorems 2 and 3] for precise statements on stability in the isotropic case which can be easily extended to the anisotropic system (1).

Roughly speaking, the first result asserts asymptotic stability for fully localized perturbations

$$v(x) = \cosh(a\xi)^{-1} w(\xi, \zeta), \quad w \in H^2(\mathbb{R}^2, \mathbb{R}^N),$$

in which a is chosen sufficiently small, $a = \mathrm{O}(|c - c_*|^{1/2})$; [6, Theorem 2]. The second result gives asymptotic stability with asymptotic phase for a class of non-localized perturbations which allow for changing the position of the corner. These perturbations are localized in any spatial direction except along the corner interface; [6, Theorem 3].

The proofs rely on a careful analysis of the spectrum of the linearization about the corner defect. The spectrum is strictly contained in the left complex half-plane, for fully localized perturbations, with an additional geometrically double eigenvalue at the origin, for the non-localized perturbations. The eigenvectors associated to this eigenvalue are the derivatives with respect to ξ and ζ of the corner defect, and therefore allow for changing the position of the corner. Nonlinear stability is then obtained by a standard fixed point argument.

5 Almost planar waves generated by instabilities

In this section we investigate different scenarios of instability in two-dimensions.

5.1 Periodically modulated corners

For the sake of simplicity in the exposition, we now restrict to the isotropic case,

$$u_t = D\Delta u + f(u) + c\partial_\zeta u, \tag{14}$$

in which $D = \mathrm{diag}\,(d^1, \dots, d^N) > 0$ is a positive, diagonal diffusion matrix. Recall that in this case the linear dispersion λ is a real-valued even function of k, and that the family $q(\varphi)$ of planar waves consists of rotations of the primary wave q_*, so that the nonlinear dispersion c is constant, $c(\varphi) = c_*$, and the flux d is convex. We now make the following assumption.

Hypothesis 5.1 (Instability) *Assume that Hypotheses 1.1 and 2.1 hold, and that*

$$\lambda''(0) > 0,\ \lambda(\pm k_*) = 0,\ \lambda'(\pm k_*) \neq 0,\ \textit{and}\ \lambda(k) > 0,\ \forall\, 0 < |k| < k_*.$$

Then the operator $\mathcal{A}_*$ has three purely imaginary eigenvalues, $\mathrm{spec}_0(\mathcal{A}_*) = \{0, \pm \mathrm{i}k_*\}$, in which 0 is algebraically double and geometrically simple, and $\pm\mathrm{i}k_*$ are both simple. In addition to the two-dimensional generalized kernel of $\mathcal{A}_*$ in Section 4 we have to consider the kernels of $\mathcal{A}_* \mp \mathrm{i}k_*$ spanned by

$$\ker\,(\mathcal{A}_* \mp \mathrm{i}k_*) = \mathrm{span}(\mathbf{e}_\pm),\ \mathbf{e}_\pm = \begin{pmatrix} r_*(\cdot) \\ \pm\mathrm{i}k_* r_*(\cdot) \end{pmatrix},$$

where $r_*(\cdot)$ is the real-valued eigenvector associated to the eigenvalue $\lambda(k_*) = 0$ of $\mathcal{L}(k_*)$.

We proceed as before and set

$$\mathbf{u} = \mathbf{q}_*^h + \eta\mathbf{e}_1^h + A\mathbf{e}_+^h + \overline{A}\mathbf{e}_-^h + \mathbf{w}^h,\ \text{with}\ P^h\mathbf{w}^h = P\mathbf{w} = 0, \tag{15}$$

in which h, η are real functions and A is a complex-valued function depending upon ξ. Since the purely imaginary eigenvalues $\pm\mathrm{i}k_*$ are complex conjugated it is more convenient to use here complex coordinates $(A, \overline{A})$, rather than real coordinates (η_1, η_2). After reduction, we find the system of ODEs

$$\begin{aligned} \eta_\xi &= \frac{2}{\lambda_{\mathrm{d}}''(0)}(c - c_*) - \frac{c_*}{\lambda_{\mathrm{d}}''(0)}\eta^2 \\ &\quad + \mathrm{O}(|c - c_*|(|c - c_*| + |\eta|^2 + |A|) + |\eta|^4 + |\eta|\,|A| + |A|^2) \end{aligned} \tag{16}$$

$$A_\xi = \mathrm{i}\alpha_*(c - c_*) + \mathrm{i}k_*A + \mathrm{O}((|c - c_*| + |\eta| + |A|)^2), \tag{17}$$

where $\alpha_* = (r_*^{\mathrm{ad}}, D^{-1}q_*') \in \mathbb{R}$, $(\cdot,\cdot)$ being the scalar product in $L^2(\mathbb{R})$.

The reduced system (16)–(17) has two equilibria

$$\eta_\pm = \mp\sqrt{2(c - c_*)/c_*} + \mathrm{O}(|c - c_*|), \quad A_\pm = \mathrm{O}(|c - c_*|),$$

corresponding to the rotated primary planar wave. In addition, the truncated system, obtained by removing the $\mathrm{O}(\cdot)$-terms, possesses a heteroclinic orbit connecting these two equilibria. This heteroclinic connection would correspond to an exterior corner in the isotropic reaction-diffusion system, in contrast to the interior corners in Theorem 4.2. However, higher order perturbations typically break this connection. For this system, it turns out that both equilibria are surrounded by a one-parameter family of periodic orbits. Then the heteroclinic orbit from the truncated system persists as a heteroclinic to one of the periodic orbits ([6, Section 4.1]), but typically *not* as a heteroclinic between the equilibria. In physical space, these heteroclinic connections correspond to exterior corners with a periodic modulation of the flat interface on either side of the corner. Similarly, symmetric exterior corners with periodic modulations on both sides of the corner exist; see Figure 4. In both cases, we expect that the minimal amplitude of the periodic structures is exponentially small in the angle of the corner for analytic kinetics [9].

Fig. 4 Plot of exterior corners with periodic modulations of the flat interface on one side (left) and both sides (right) of the corner.

Remark 5.2 (The onset of transverse instability) Further types of almost planar waves can be constructed by considering the onset of transverse instability for systems depending upon an additional parameter μ. This situation has been analyzed in the isotropic case in [6, Section 4.2]. It turns out that in this case the reduced system is a perturbed Kuramoto-Sivashinsky equation possessing heteroclinic and homoclinic orbits which correspond to both interior and exterior corners, and steps.

5.2 Existence of steps

In this section we consider a different instability scenario by studying stability boundaries between stable and unstable directions of propagation φ. Roughly speaking, we assume that the planar waves $q(\varphi)$ develop an instability for small transverse wavenumbers k while φ crosses a critical angle φ_*. Clearly, this scenario is particular to anisotropic media since stability properties in isotropic media do not depend on the direction of propagation.

Hypothesis 5.3 (Lateral instability) *Assume that Hypotheses 1.1 and 2.1 hold, and denote by $\lambda(k;\varphi)$ the continuation of the simple eigenvalue $\lambda = 0$ of $\mathcal{L}_0$ for small k and φ. We make the following assumption:*

$$\lambda(0;\varphi) = 0, \quad \lambda'(0;\varphi_*) = \lambda''(0;\varphi_*) = 0, \quad \lambda'''(0;\varphi_*) \neq 0, \quad \partial_\varphi \lambda''(0;\varphi_*) > 0,$$

where $'$ denotes differentiation with respect to k.

We follow the same strategy of proof as before, with the difference that we allow here for nearby directions $n \sim n_*$, by taking the angle φ as a second parameter in the dynamical system which is now of the form

$$\mathbf{u}_\xi = \mathcal{A}(c,\varphi)\mathbf{u} + \mathcal{F}(\mathbf{u}). \tag{18}$$

The reduced system will then describe almost planar interfaces for speeds c close to c_* and angles φ close to φ_*.

The purely imaginary spectrum of the linearized operator $\mathcal{A}_*$ consists of the eigenvalue 0 which is geometrically simple, but now algebraically triple. We then decompose

$$\mathbf{u} = \mathbf{q}_*^h + \eta_1 \mathbf{e}_1^h + \eta_2 \mathbf{e}_2^h + \mathbf{w}^h, \text{ with } P^h \mathbf{w}^h = P\mathbf{w} = 0, \tag{19}$$

in which h, η_1, η_2 are real functions depending upon ξ. The computation of the reduced system is in this case more involved, but we can still find the lowest order part of this system explicitly and finally reduce it to the second order ODE

$$\begin{aligned} \eta_{1\xi\xi} \quad = \quad & -\frac{6}{\mathrm{i}\lambda'''(0;\varphi_*)}\Big((c-c_*) - \frac{c_*}{2}(\varphi-\varphi_*)^2 + c_*(\varphi-\varphi_*)\eta_1 \\ & \qquad -\frac{d''(\varphi_*)}{2}\eta_1^2 - \frac{\partial_\varphi \lambda''(0;\varphi_*)}{2}\eta_1\eta_{1\xi}\Big) \\ & + \mathrm{O}(|c-c_*|(|c-c_*| + |\varphi-\varphi_*| + |\eta_1| + |\eta_{1\xi}|) \\ & \qquad + |\eta_{1\xi}|^2 + (|\varphi-\varphi_*| + |\eta_1|)^3 + |\eta_{1\xi}|(|\varphi-\varphi_*|^2 + |\eta_1|^2)). \end{aligned}$$

We restrict our interest to homoclinic orbits of this second order equation which correspond to step solutions of the reaction-diffusion system. We first notice that the truncated equation

$$\eta_{1\xi\xi} = -\frac{6}{\mathrm{i}\lambda'''(0;\varphi_*)}\left((c-c_*) - \frac{c_*}{2}(\varphi-\varphi_*)^2 + c_*(\varphi-\varphi_*)\eta_1 - \frac{d''(\varphi_*)}{2}\eta_1^2\right)$$

does possess homoclinic orbits for $\varphi = \varphi_*$, and any $c > c_*$ (resp. $c < c_*$) if the flux is convex (resp. concave). Our purpose is to show that these orbits persist for the full equation as a family $(\eta_1^*(\varepsilon); c(\varepsilon), \varphi(\varepsilon))$, with small ε. As opposed to many systems where the KdV equation provides the leading order approximation, the present system does not possess reversibility, Hamiltonian structure, or a Galilei invariance, such that parameters are necessary to show persistence of the homoclinic orbit. The key ingredient to the proof is a sequence of transformations and scalings, which isolate a linear damping term $\eta_{1\xi}$ as the first order correction. Since this damping term unfolds the homoclinic transversely, we can conclude persistence using the implicit function theorem; see [7, Section 6] for details. As a result, we find the following theorem, which asserts existence of steps along a smooth curve in the parameter space (c, φ).

Theorem 5.4 (Existence of steps [7, Theorem 2]) *Assume that Hypothesis 5.3 holds and that the flux d is either convex or concave, $d''(\varphi_*) \neq 0$. Then there exists a smooth curve*

$$\varepsilon \mapsto (c(\varepsilon), \varphi(\varepsilon)) = (c_* + \mathrm{O}(\varepsilon^4), \varphi_* + \mathrm{O}(\varepsilon^2))$$

defined for small $\varepsilon > 0$, such that the system (1) possesses a family of steady solutions $(u_\varepsilon)_{\varepsilon>0}$ with ε small which are steps propagating with speed $c(\varepsilon)$ in the direction $n(\varphi(\varepsilon))$.

Remark 5.5 (Stability of steps) From the proof, it follows that the steps are asymptotic to stable planar interfaces. In particular, the essential spectrum of the linearization is marginally stable and can be pushed into the negative left half plane by means of exponential weights. This is in complete analogy with the Korteweg-de Vries equation, which can be formally derived as a modulation equation in the present situation. We conjecture that the steps are actually spectrally stable, that is, the point spectrum of the linearized operator is contained in the closed left half plane. Again, this is suggested by the KdV approximation. Although higher-order terms do not preserve the symmetries of the KdV equation, we suspect that the spectral picture of the KdV equation persists.

6 Holes in oscillatory wave propagation

So far, the analysis of almost planar waves showed the existence of three types of corner defects: interior and exterior corners, and steps. Holes appear to be excluded in these simplest scenarios. We now present a scenario, where holes do exist in isotropic media. However, the underlying planar wave will be assumed to propagate in an oscillatory fashion; [6, Section 5].

Consider again the isotropic system

$$u_t = D\Delta u + f(u) + c\partial_\zeta u, \tag{20}$$

in the plane $(\xi, \zeta) \in \mathbb{R}^2$. We will be interested in *modulated traveling waves*,

$$u(\xi, \zeta, t) = q(\xi, \zeta, \omega t),$$

in which q is 2π-periodic in its third argument. We retain the concept of almost planar waves close to a known planar (modulated) wave $u(\xi, \zeta, t) = q_*(\zeta, \omega t)$, which connects two homogeneous equilibria $q_\pm$ as $\zeta \to \pm\infty$, and is 2π-periodic in its second argument. Modulated planar waves solve

$$D\partial_{\zeta\zeta} u + c\partial_\zeta u + f(u) - \omega\partial_t u = 0, \tag{21}$$

for some speed $c = c_*$ and frequency $\omega = \omega_*$, where ωt is replaced by t. Almost planar (modulated) interfaces are constructed as bounded solutions to

$$D\Delta u + c\partial_\zeta u + f(u) - \omega\partial_t u = 0, \tag{22}$$

which are 2π-periodic in t. In the following, we shall focus on existence of holes; see also [8, Section 7] for a discussion of other type of corner defects in this context.

6.1 Existence of holes

The starting point is again a hypothesis on existence of a planar wave connecting two homogeneous equilibria, now a modulated planar wave, then followed by assumptions on linear stability.

Hypothesis 6.1 (Existence) *We assume that there exist positive constants c_*, ω_*, and homogeneous states $q_\pm$ such that there exists an ξ-independent planar modulated-wave solution $q_*(\zeta, t)$ of (22) which is 2π-periodic in t, with $\partial_t q_* \not\equiv 0$, and which connects q_- and q_+, that is,*

$$q_*(\zeta, t) \to q_+ \text{ for } \zeta \to +\infty, \quad q_*(\zeta, t) \to q_- \text{ for } \zeta \to -\infty,$$

uniformly in t.

Next, consider the linearized operator

$$\mathcal{M}_* u = D\Delta u + c_*\partial_\zeta u + f'(q_*)u - \omega_*\partial_t u, \tag{23}$$

and its Fourier conjugates

$$\mathcal{M}(k)u = D(\partial_{\zeta\zeta} - k^2)u + c_*\partial_\zeta u + f'(q_*)u - \omega_*\partial_t u, \tag{24}$$

on the Hilbert spaces $L^2(\mathbb{R}^2 \times S^1, \mathbb{R}^N)$ and $L^2(\mathbb{R} \times S^1, \mathbb{R}^N)$, respectively. We consider the simplest situation of a stable modulated front.

Hypothesis 6.2 (Transverse stability) *We assume that*

(i) the spectrum of $\mathcal{M}_0 := \mathcal{M}(0)$ is contained in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\} \cup \{0\}$;

(ii) $\mathcal{M}_0$ is Fredholm with index zero and has a two-dimensional kernel and generalized kernel spanned by the partial derivatives $\partial_\zeta q_*$ and $\partial_t q_*$ of the modulated wave q_*;

(iii) the spectra of $\mathcal{M}(k)$, for $k \neq 0$ are strictly contained in $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda < 0\}$;

(iv) the 2×2 matrix $\Lambda_\mathrm{d}(k)$ with $\Lambda_\mathrm{d}(0) = \Lambda'_\mathrm{d}(0) = 0$, representing the smooth continuation for $k \sim 0$ of the action of $\mathcal{M}_0$ on its kernel, satisfies $\operatorname{Re}\operatorname{spec}\Lambda''_\mathrm{d}(0) < 0$.

The strategy of the existence proof is essentially the same as before, but the reduction procedure and the analysis of the reduced system are much more involved; see [6, Section 5].

We set $\mathbf{u} = (u, v)^T$ and rewrite the equation (22) as a dynamical system

$$\mathbf{u}_\xi = \mathcal{A}(c, \omega)\mathbf{u} + \mathcal{F}(\mathbf{u}), \tag{25}$$

on the Hilbert space $Y = (H^{1,\frac{1}{2}} \times L^2)(\mathbb{R} \times S^1, \mathbb{R}^N)$. The linear and nonlinear parts of (25) are given by

$$\mathcal{A}(c,\omega) = \begin{pmatrix} 0 & \mathrm{id} \\ -\partial_{\zeta\zeta} - D^{-1}c\partial_\zeta + D^{-1}\omega\partial_t & 0 \end{pmatrix}, \quad \mathcal{F}(\mathbf{u}) = \begin{pmatrix} 0 \\ -D^{-1}f(u) \end{pmatrix}.$$

Hypothesis 6.1 shows that (25) has a *two-parameter* family of equilibria

$$\mathbf{q}_*^{h,\tau} = \begin{pmatrix} q_*^{h,\tau}(\cdot,\cdot) \\ 0 \end{pmatrix} = \begin{pmatrix} q_*(\cdot + h, \cdot + \tau) \\ 0 \end{pmatrix}, \tag{26}$$

due to translation invariance in both ζ and t. From Hypothesis 6.2 we conclude that the spectrum of the linearization $\mathcal{A}_*$ of (25) about $\mathbf{q}_*^{0,0}$, satisfies

$$\operatorname{spec}\mathcal{A}_* \cap \{|\operatorname{Re}\lambda| \leq \epsilon\} = \{0\},$$

in which the origin is an eigenvalue with geometric multiplicity two and algebraic multiplicity four. The kernel and generalized kernel of $\mathcal{A}_*$ are spanned by

$$\ker\mathcal{A}_* = \operatorname{span}(\mathbf{e}_{0\zeta}, \mathbf{e}_{0t}), \quad \mathbf{e}_{0\zeta} = \begin{pmatrix} \partial_\zeta q_* \\ 0 \end{pmatrix}, \ \mathbf{e}_{0t} = \begin{pmatrix} \partial_t q_* \\ 0 \end{pmatrix},$$

and

$$\operatorname{gker}\mathcal{A}_* = \operatorname{span}(\mathbf{e}_{0\zeta}, \mathbf{e}_{0t}, \mathbf{e}_{1\zeta}, \mathbf{e}_{1t}), \quad \mathcal{A}_*\mathbf{e}_{1\zeta} = \mathbf{e}_{0\zeta}\ \mathcal{A}_*\mathbf{e}_{1t} = \mathbf{e}_{0t}.$$

We construct the spectral projection P onto the generalized of $\mathcal{A}_*$, and then make the Ansatz

$$\mathbf{u} = \mathbf{q}_*^{h,\tau} + \eta\mathbf{e}_{1\zeta}^{h,\tau} + \rho\mathbf{e}_{1t}^{h,\tau} + \mathbf{w}^{h,0}, \text{ with } P^{h,\tau}\mathbf{w}^{h,0} = P^{0,\tau}\mathbf{w}^{0,0} = 0, \tag{27}$$

where h, τ, η, and ρ are functions depending upon ξ. Again the equations for h and τ decouple, and a now more delicate reduction procedure gives us a reduced system for η and ρ.

Holes correspond to homoclinic orbits to the origin in the reduced system. It turns out that the origin is an equilibrium of the reduced system only if $c = c_*$ and $\omega = \omega_*$ when the reduced system reads

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{pmatrix}\eta\\ \rho\end{pmatrix} = c_*\begin{pmatrix} d_{11} & d_{21}\\ d_{12} & d_{22}\end{pmatrix}\begin{pmatrix}\frac{1}{2}\eta^2 - \alpha_1\rho^2\\ \eta\rho - \alpha_2\rho^2\end{pmatrix} + \mathrm{O}((|\eta| + |\rho|)^3), \tag{28}$$

in which

$$\begin{pmatrix} d_{11} & d_{12}\\ d_{21} & d_{22}\end{pmatrix} = -2(\Lambda_{\mathrm{d}}''(0))^{-1},$$

and

$$\alpha_1 = \frac{\langle q_{*\zeta}^{\mathrm{ad}}, \partial_{tt}q_*\rangle d_{22} - \langle q_{*t}^{\mathrm{ad}}, \partial_{tt}q_*\rangle d_{21}}{c_*(d_{11}d_{22} - d_{12}d_{21})}, \quad \alpha_2 = \frac{\langle q_{*t}^{\mathrm{ad}}, \partial_{tt}q_*\rangle d_{11} - \langle q_{*\zeta}^{\mathrm{ad}}, \partial_{tt}q_*\rangle d_{12}}{c_*(d_{11}d_{22} - d_{12}d_{21})},$$

where $\langle\cdot,\cdot\rangle$ represents the scalar product in $L^2(\mathbb{R}\times S^1, \mathbb{R}^N)$. To leading order we have a quadratic system of ODEs in which the origin is typically an isolated equilibrium. For certain values of the coefficients such systems possess homoclinic solutions which approach the origin along ray solutions of the form

$$\eta(\xi) = -\frac{\eta_*}{\xi}, \quad \rho(\xi) = -\frac{\rho_*}{\xi};$$

see Figure 5. In particular, they decay algebraically, $(\eta,\rho)(\xi) = \mathrm{O}(1/|\xi|)$ as $|\xi| \to \infty$. For the

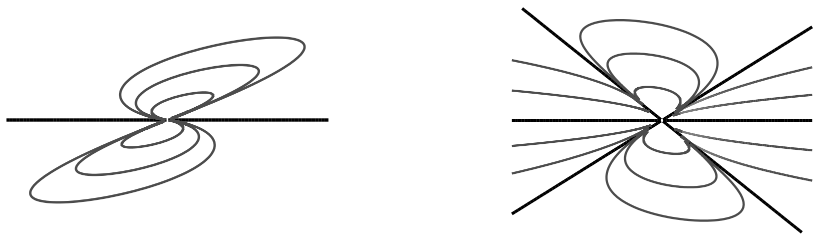


Fig. 5 Typical homoclinic orbits of the quadratic system in the case of one ray solution (left) and three ray solutions (right).

reduced system (28) the construction of such homoclinic orbits relies upon geometric blow-up techniques. We refer to [6, Section 5] for the proof of the following result.

Theorem 6.3 (Existence of holes [6, Proposition 5.4]) *Assume that the coefficients α_1 and α_2 of the reduced system (28) satisfy $\alpha_2^2 < 2\alpha_1$. There is $\varepsilon_0 > 0$ such that (28) possesses two families of homoclinic orbits,*

$$(\eta_\varepsilon^\pm(\xi), \rho_\varepsilon^\pm(\xi)) = (\pm\varepsilon\eta_o(\pm\varepsilon\xi), \pm\varepsilon\rho_o(\pm\varepsilon\xi)) + \mathrm{O}(\varepsilon^2), \tag{29}$$

for $\varepsilon \in (0, \varepsilon_0)$, in which (η_o, ρ_o) decay like $\mathrm{O}(1/|\xi|)$ as $|\xi| \to \infty$. For $\alpha_2^2 > 2\alpha_1$ there are no nontrivial, small, bounded solutions to (28).

Consequently, there is an open class of reaction-diffusion systems which possess a one-parameter family of holes close to a stable planar modulated wave with the same speed c_* and frequency ω_*.

We point out that the algebraic decay of η and ρ generates a logarithmic divergence of the position h and the temporal phase τ of the interface for the holes in this theorem.

6.2 Stability properties and quadratic systems of conservation laws

The question of stability of holes is widely open. However, we can get some insight into this question by looking at approximating model equations. In [6, Appendix B], we derived a system of viscous conservation laws which formally describes the two-dimensional dynamics of planar modulated waves in the reaction-diffusion system (20) for perturbations with slowly varying temporal phase in a long-wave regime. Written in the variables η and ρ, this model system is a time-dependent version of the reduced equation (28),

$$\begin{pmatrix} \eta_t \\ \rho_t \end{pmatrix} = -\frac{1}{2}\Lambda_{\mathrm{d}}''(0) \begin{pmatrix} \eta_{\xi\xi} \\ \rho_{\xi\xi} \end{pmatrix} - \begin{pmatrix} \left(\frac{1}{2}\eta^2 - \alpha_1\rho^2\right)_\xi \\ \left(\eta\rho - \alpha_2\rho^2\right)_\xi \end{pmatrix}. \qquad (30)$$

Interestingly, the algebraic condition on the coefficients of the reduced system in Theorem 6.3, which ensures the existence of homoclinic orbits, turns out to be the condition under which the system in the zero-viscosity limit has complex characteristics: the conservation law is ill-posed, elliptic rather than well-posed, hyperbolic.

Though much simpler than the question of stability of holes, even the stability of homoclinic solutions in (30) is not completely understood. So far, the only answer to this question concerns the special case of equal diffusion coefficients, $(d_{ij}) \sim \mathrm{id}$, and shows that despite the highly unstable inviscid limit, localized solutions to (30) are asymptotically stable [8]. In this case, the system (30) turns out to be equivalent to the complex continuation of Burgers equation

$$Z_t = Z_{\xi\xi} - ZZ_\xi, \quad Z(\xi, t) \in \mathbb{C},$$

with homoclinic orbits explicitly given by

$$Z_*(\xi) = -\frac{2}{\xi + \mathrm{i}a + b}, \quad a \in \mathbb{R}^*,\ b \in \mathbb{R}.$$

While Bessel functions theory allows to solve the eigenvalue problem and obtain spectral stability for L^2-perturbations, a complex continuation of the classical Hopf-Cole transformation gives nonlinear stability for perturbations with zero mass. In particular, this result shows that holes may well provide islands of stability in ill-posed elliptic conservation laws.

7 Inhomogeneities

In this section, we briefly illustrate this approach in the case of front propagation in the presence of inhomogeneities. Envision, for instance, the propagation of a planar flame front in a direction perpendicular to a strip in the plane, where the speed of propagation of the front differs. We may think of the slower spread of a forest fire along cool and moist river shores or the faster spread of a combustion front along a catalyzer plate. The model problem that we investigate is

$$u_t = D\Delta u + f(u) + \varepsilon g(u; x_1), \qquad (31)$$

where $|g(u;x_1)| = \mathrm{o}(x_1^{-1})$, uniformly for bounded u.

The center manifold reduction in Section 3 can be adapted to this inhomogeneous setting. Assuming that the transverse group velocity vanishes it yields (after a suitable scaling of h, η and ξ) the reduced equation

$$h' = \eta, \quad \eta' = \tilde{c} - \eta^2 + \varepsilon\tilde{g}(\xi) + \text{h.o.t.}, \tag{32}$$

in which $\tilde{c}$ and $\tilde{g}$ can be computed from $c - c_*$ and g. This equation is easy to discuss analytically in the extended phase space $(h,\eta,\xi) \in \mathbb{R}^2 \times \overline{\mathbb{R}}$, where we can compactify ξ since $g, \tilde{g} \to 0$, for $|\xi| \to \infty$. The result however can simply be deduced from the conjugated linear equation: equation (32) is a Riccati equation with associated linear equation for $\psi = \mathrm{e}^h$,

$$\psi'' = \varepsilon\tilde{g}(\xi)\psi + \tilde{c}\psi. \tag{33}$$

Then, for $(\varepsilon \int \tilde{g}) < 0$, there exists a unique positive eigenfunction to this eigenvalue problem, which yields an exterior corner. The eigenvalue $\sqrt{-\varepsilon \int \tilde{g}} + \mathrm{O}(\varepsilon)$ determines the wavespeed and the angle of the exterior corner. For $(\varepsilon \int \tilde{g}) > 0$, no such eigenfunction exists. In this case, the embedded eigenvalue at the edge of the spectrum is a resonance pole, which generates holes for $\tilde{c} = 0$, with asymptotically flat interface diverging as $|\xi| \to \infty$.

Another interesting situation occurs when the inhomogeneity travels, or when the interface possesses a nonzero transverse group velocity. In this case, the center manifold is one-dimensional, the reduced equation simply reads

$$h' = \varepsilon\tilde{g}(\xi) + \text{h.o.t.}.$$

Then, $(\varepsilon \int \tilde{g})$ this time determines the jump in the position, $h(+\infty) - h(-\infty)$, and the interface typically forms a step across the interface.

We summarize these findings in the following theorem.

Theorem 7.1 (Exterior corners and steps in inhomogeneous media) *Assume that the integral of the reduced inhomogeneity $\tilde{g}$ is positive and that the transverse group velocity vanishes. We then have unique exterior corners for $\varepsilon < 0$ and unique hole solutions with asymptotically horizontal interface for $\varepsilon > 0$. In case of nonzero group velocity, the interface forms an upward and downward step across the inhomogeneity, respectively.*

We conclude with a numerical illustration of these results. We simulate

$$u_t = \triangle u + \frac{1}{\delta}u(1-u)(u - \frac{v+a}{b}), \quad v_t = 0.01\triangle v + u - v + \varepsilon g, \tag{34}$$

with $a = 0.6, b = -0.05, \delta = 0.08, |x_j| \leq 90, \varepsilon = 0.1$, and

$$g(x) = \frac{1}{1+x_1^2} \quad \text{or} \quad g(x) = \frac{1}{1+(x_1-x_2-95)^2}.$$

This system is a variant of the FitzHugh-Nagumo equation [1] and can be considered a caricature model for excitable and oscillatory media such as the Belousov-Zhabotinsky equation. Light-sensitive variants of the BZ-reaction allow experimentalists to change recovery speed and excitability properties of the medium locally in space. The resulting wave breaking has been explained in terms of geometric optics in [12]. Our results allow for a systematic and rigorous approach to discontinuities in wave fronts such as corners and steps, generated by such spatial inhomogeneity of the medium.

The system (34) possesses a family of stable planar wave trains. Figure 6 illustrates the four types of corners in this setup. The inhomogeneity is located on a centered, vertical line, in the two left examples, and along the diagonal in the right-hand picture. Along the bottom boundary, we created waves with an additional local inhomogeneity. Neumann boundary conditions are imposed along all sides of the square.

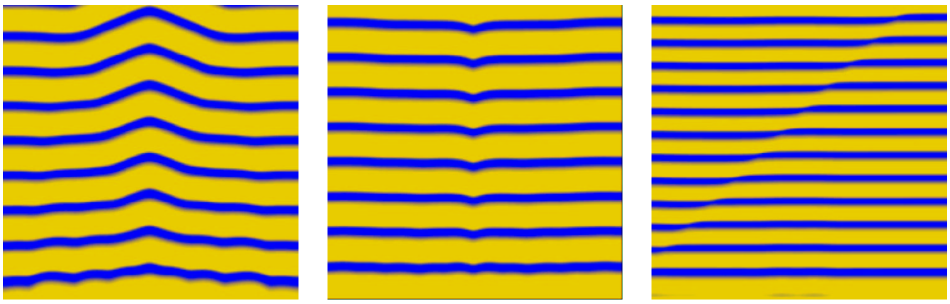


Fig. 6 Numerical simulations of wavetrains in media with a line inhomogeneity. In the left-hand picture, the line inhomogeneity acts as a source, thus generating an exterior corner. Also visible in the picture are interior corners at the interface of the exterior corner and the horizontal wave trains. In the middle picture the line inhomogeneity weakly absorbs waves, thus creating holes with the typical slow logarithmic divergence of the shape of the wave. In the right-hand picture, the line inhomogeneity travels with a constant speed to the right in a frame moving upwards with the interface, thus creating a step.

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References

[1] D. Barkley, *A model for fast computer simulation of waves in excitable media*, Physica D **49** (1991), 61–70.

[2] A. Bonnet and F. Hamel, *Existence of nonplanar solutions of a simple model of premixed Bunsen flames*, SIAM J. Math. Anal. **31** (1999), 80–118.

[3] J.W. Cahn and D.W. Hoffmann. *A vector thermodynamics for anisotropic interfaces. 1. Fundamentals and applications to plane surface junctions*, Surf. Sci. **31** (1972), 368–388.

[4] J.W. Cahn and D.W. Hoffmann. *A vector thermodynamics for anisotropic interfaces. 2. Curved and faceted surfaces*, Acta Metall. **22** (1974), 1205–1214.

[5] F. Hamel, R. Monneau, and J.-M. Roquejoffre, *Existence and qualitative properties of multidimensional conical bistable fronts*, Disc. Cont. Dyn. Systems, to appear.

[6] M. Haragus and A. Scheel, *Corner defects in interface propagation*, Ann. Inst. H. Poincare (C) Anal. Non Lineaire, to appear.

[7] M. Haragus and A. Scheel, *Almost planar waves in anisotropic media*, Comm. Partial Differential Equations, to appear.

[8] M. Haragus and A. Scheel, *Stable viscous shocks in elliptic conservation laws*, preprint.

[9] E. Lombardi, *Oscillatory integrals and phenomena beyond all algebraic orders. With applications to homoclinic orbits in reversible systems*, Lecture Notes in Mathematics **1741**, Springer-Verlag, Berlin, 2000.

[10] A. Mielke, *Reduction of quasilinear elliptic equations in cylindrical domains with applications*, Math. Methods Appl. Sci. **10** (1988), 51–66.

[11] H. Ninomiya and M. Taniguchi, *Existence and global stability of traveling curved fronts in the Allen-Cahn equations*, J. Differential Equations, to appear.

[12] J. Sainhas and R. Dilao, *Wave Optics in Reaction-Diffusion Systems*, Phys. Rev. Lett. **80**, (1998) 5216–5219.

[13] B. Sandstede, A. Scheel, and C. Wulff, *Dynamics of spiral waves on unbounded domains using center-manifold reduction*, J. Differential Equations **141** (1997), 122–149.

[14] A. Scheel, *Radially symmetric patterns of reaction-diffusion systems*, Mem. Amer. Math. Soc. **165** (2003).

[15] E. Schöll, *Nonlinear Spatio-Temporal Dynamics and Chaos in Semiconductors*, Cambridge Nonlinear Science Series **10**, Cambridge University Press, 2000.

[16] A. I. Volpert, V. A. Volpert, and V. A. Volpert, *Traveling wave solutions of parabolic systems*, Translations of Mathematical Monographs **140**, American Mathematical Society, Providence, RI, 1994.