Solitary waves and their linear stability in weakly coupled KdV equations

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Abstract

We consider a system of weakly coupled KdV equations developed initially by Gear & Grimshaw to model interactions between long waves. We prove the existence of a variety of solitary wave solutions, some of which are not constrained minimizers. We show that such solutions are always linearly unstable. Moreover, the nature of the instability may be oscillatory and as such provides a rigorous justification for the numerically observed phenomenon of "leapfrogging."

1 Introduction

The Korteweg-de Vries (KdV) equation, $u_t+(u_{xx}+u^2)_x=0$, is a well-known model for many processes involving the evolution of long waves. More specifically, Korteweg & de Vries and Boussinesq derived the KdV equation to model the behavior of surface water waves in a flat-bottomed canal.[1] (u is roughly proportional to the surface elevation of the water.) The KdV equation famously possesses solitary wave solutions of the form $u(x,t)=\frac{3c}{2}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct-x_0)\right)$ where the wave speed c is positive.

Gear & Grimshaw in [9] derived a system of coupled KdV equations to model interactions of long waves, for example in a stratified fluid. Specifically, their model is

$$\begin{aligned} u_t+\left(u_{xx}+u^2+\epsilon_1 v^2+\epsilon_2 uv+\epsilon_3 v_{xx}\right)_x &=0\\ c_1 v_t+\left(c_2 v+v_{xx}+v^2+c_3(\epsilon_1 uv+\epsilon_2 u^2+\epsilon_3 u_{xx})\right)_x &=0. \end{aligned} \tag{1}$$

Here, c_j and ϵ_j, $j=1,2,3$, are real valued constants and u and v are the displacement from horizontal of the fluid interfaces. Liu, Kubota & Ko developed an alternate set of equations for like phenomena in [14]. The primary difference between the two models is that the Liu-Kubota-Ko system utilizes a Fourier multiplier operator in lieu of second derivatives and the coupling is strictly linear.

[1] The KdV equation has subsequently been discovered to model waves in atomic lattices and in plasmas, possess the remarkable property of "complete integrability" and have applications in geometry.

Equations similar to (1) also arise in the study of two-dimensional atomic lattices and head-on collisions of solitary water waves (see [20]). We will consider systems of the following type, of which (1) is a specific example:

$$\begin{aligned} u_t + \left(K_1 u + F_1'(u) + \epsilon \left(K_3 v + \partial_u H(u,v)\right)\right)_x &= 0 \\ v_t + \left(K_2 v + F_2'(v) + \epsilon \left(K_3 u + \partial_v H(u,v)\right)\right)_x &= 0. \end{aligned} \tag{2}$$

Here $(u,v)^t \in \mathbf{R}^2$, $x \in \mathbf{R}$ and $t \geq 0$. The functions F_1 and F_2 are C^∞ maps from $\mathbf{R}$ to $\mathbf{R}$ and the coupling function $H(u,v)$ is a C^∞ map from $\mathbf{R}^2$ to $\mathbf{R}$. We require that all first and second derivatives of F_1, F_2 and H are zero when evaluated at zero. The maps K_1, K_2 and K_3 will be the constant coefficient differential operators

$$K_1 = \partial_x^2, \; K_2 = c_1 + c_2\partial_x^2, \text{ and } K_3 = c_3 + c_4\partial_x^2.$$

We require that c_2 is positive. The operators, since they only include even derivatives, are self-adjoint on L^2 (with the standard inner product $(f,g)_{L^2} = \int f(x)g(x)dx$).[2]

Remark 1. *We could allow the K_j to be more general self-adjoint operators, such as those Fourier multiplier operators which appear in the Liu-Kubota-Ko model or constant coefficient differential operators which contain only even derivatives. There are a multitude of minor complications that arise in these cases which occlude our methods and provide no additional interesting insight into the problems we consider.*

In this document we examine the existence and linear stability of solitary wave solutions to (2) when the coupling is weak, *i.e.* when ϵ is close to zero. Our approach is perturbative; an enormous amount of information is known about the existence and stability of solitary waves in single generalized KdV equations and, as a consequence, we have a more or less complete understanding of solitary wave solutions in the uncoupled ($\epsilon = 0$) problem. We are able to use a Liapunov-Schmidt analysis to determine the existence of a variety of solitary wave solutions for weak coupling. Specifically we prove the existence of four different types of solitary waves. The first type is $O(1)$ in the u component and $O(\epsilon)$ in the v component. The second is the same as the first but with the roles of u and v switched. These solutions have been discovered previously using variational means by Albert & Linares (in [1]) and Bona & Chen (in [4]). The third (and more interesting) type of solution is $O(1)$ in both components simultaneously. The u and v components are even functions on their own and share a common center of mass. We have also determined a criterion for the existence of a fourth type of solitary wave which is $O(1)$ in both components but the components do not share a common center.[3] We show that the first two types of solitary waves are orbitally stable by an appeal to the abstract theory of Grillakis, Shatah & Strauss ([11] [12]). We compute the spectrum of the linearization about the last two types using reduction methods and perturbation theory. It turns out that the third and fourth types are always linearly unstable, though the nature of the instability comes in two distinct forms.

The remainder of this paper is organized as follows. In Section 2, we lay out the problem in greater detail and discuss our main results. In Sections 3 and 4 and in the Appendix we prove the existence of solitary waves. We discuss linear stability in Sections 5 and 6. In Section 7 we present the results of some numerical simulations which demonstrate our results.

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[2] All integrals which appear in this document are taken over all of $\mathbf{R}$, unless otherwise specified.

[3] Results similar to these have been established for some coupled dissipative systems, see [7] for example

2 Preliminaries and the main results

We now remark on several important features of (2). System (2) is a hamiltonian partial differential equation and can be rewritten as

$$\mathbf{u}_t = \partial_x E'_\epsilon[\mathbf{u}],$$

where $\mathbf{u} = (u, v)^t$ and

$$\begin{aligned} E_\epsilon[\mathbf{u}] \equiv & -\int \left(\frac{1}{2}uK_1u + \frac{1}{2}vK_2v + F_1(u) + F_2(v)\right) dx \\ & -\epsilon \int (uK_3v + H(u,v))\, dx. \end{aligned}$$

We refer to $E_\epsilon[\mathbf{u}]$ as the *energy* of the solution and this quantity is a constant of the motion. E'_ϵ is the gradient of E taken with respect to the usual inner product on $L^2 \times L^2$

$$\langle \mathbf{f}, \mathbf{g} \rangle = \int \mathbf{f}(x) \cdot \mathbf{g}(x) dx.$$

The symplectic structure is generated by ∂_x.

System (2) is invariant under spatial translations. That is, if $\mathbf{u}(x,t)$ (where $\mathbf{u} = (u,v)^t$) is a solution of (2) then so is $\mathbf{u}(x - x_0, t)$. Note that this translation occurs simultaneously in the u and v components. This invariance gives rise to the conserved quantity

$$M[\mathbf{u}] \equiv \int \frac{1}{2}\left(u^2 + v^2\right) dx$$

which we call the *momentum*. A simple but important observation about the uncoupled problem is that it is translation invariant in each component independently. As a consequence if $\epsilon = 0$ and $(u(x,t), v(x,t))^t$ is a solution of (2), then so is $(u(x - x_0, t), v(x + x_0, t))^t$. It is the breaking of this symmetry which is at the heart of the existence and stability of solitary waves when $\epsilon \neq 0$. We will call this symmetry the *separation invariance* of the uncoupled problem.

2.1 Existence

Solitary waves are solutions of the form

$$\mathbf{u}(x,t) = \mathbf{q}_{c,\epsilon}(x - ct) = \begin{pmatrix} q(x - ct) \\ r(x - ct) \end{pmatrix}.$$

It is important to notice here that we require that *the wave has the same speed in each component.* Inserting the above *Ansatz* into (2), we find that the functions q and r satisfy the system of ordinary differential equations

$$\begin{aligned} -cq + K_1q + F_1'(q) + \epsilon\,(K_3r + \partial_u H(q,r)) &= 0 \\ -cr + K_2r + F_2'(r) + \epsilon\,(K_3q + \partial_v H(q,r)) &= 0. \end{aligned} \tag{3}$$

The above can be rewritten as

$$E'_\epsilon[\mathbf{q}_{c,\epsilon}] + cM'[\mathbf{q}_{c,\epsilon}] = 0$$

which is to say that the profile, $\mathbf{q}_{c,\epsilon}$, of a solitary wave solution is a critical point of E_ϵ under the constraint that M is a constant. (The wave-speed c is the Lagrange multiplier.)

Previous studies (*e.g.* [4] [1]) on the existence and stability of solitary waves to equations like (2) have relied heavily on this fact. These methods are powerful in the sense that they do not require the coupling to be small and can be used to prove full nonlinear stability of the solitary waves. The existence results utilize the powerful "concentration compactness" tools of Lions [13]. However, they typically locate *global* minimizers. As we shall demonstrate, there are numerous types of solitary wave solutions to (2), several of which are *not* minimizers of the energy—hence our perturbative approach.

In the case $\epsilon = 0$, (2) is simply an uncoupled pair of generalized KdV equations and, given reasonable assumptions on K_1, K_2, F_1 and F_2, each has a solitary wave solution. These functions are solutions of the following second order differential equations

$$\begin{aligned} -cq + q'' + F_1'(q) &= 0 \\ -(c - c_1)r + c_2 r'' + F_2'(r) &= 0 \end{aligned} \tag{4}$$

which are homoclinic at zero.

The existence of such solutions can be determined by a fairly straight-forward phase plane analysis which relies very strongly on the nature of the nonlinearities. Homoclinic solutions are typically even (up to a translation) and this is another property we exploit. We make the following hypothesis:

Hypothesis 2. *There exist non-zero, even C^∞ functions $q_{c,0}(y)$ for c in the interval I_1 and $r_{c,0}(y)$ for c in the interval I_2 which are solutions to the equations in (4) (respectively) which are homoclinic to zero. We assume that $I_0 = I_1 \cap I_2$ contains an nonempty open interval.*

Remark 3. *The implicit function theorem can be applied to conclude that the dependence of the functions $q_{c,0}$ and $r_{c,0}$ on c is C^∞.*

Since the solutions are homoclinic, they will behave at spatial infinity like the linear problems $-cf + f'' = 0$ and $-(c - c_1)f + c_2 f'' = 0$. If $c > 0$ in the first equation and if $c > c_1$ in the second then $f = 0$ is an hyperbolic equilibrium and as a result solutions which tend to zero do so at an exponential rate. If $c = 0$ or $c = c_1$ then there are choices for the nonlinearities which give rise to solitary wave solutions which decay only algebraically. In our analysis we will need the exponential decay of the solutions and so we shall assume this as well. That is we assume $I_1 \subset (0, \infty)$ and $I_2 \subset (c_1, \infty)$.

Hypothesis 2 along with the separation invariance of the problem when $\epsilon = 0$ show that there exist four distinct types of solitary wave solutions with speed c to (2) in the uncoupled case. These are:

$$\mathbf{u}(x,t) = \mathbf{t}_{c,0}(x - ct) \equiv \begin{pmatrix} q_{c,0}(x - ct) \\ 0 \end{pmatrix} \tag{5}$$

$$\mathbf{u}(x,t) = \mathbf{b}_{c,0}(x - ct) \equiv \begin{pmatrix} 0 \\ r_{c,0}(x - ct) \end{pmatrix} \tag{6}$$

$$\mathbf{u}(x,t) = \mathbf{p}_{c,0}(x - ct) \equiv \begin{pmatrix} q_{c,0}(x - ct) \\ r_{c,0}(x - ct) \end{pmatrix} \tag{7}$$

and

$$\mathbf{u}(x,t) = \mathbf{p}_{c,0,x_1}(x - ct) \equiv \begin{pmatrix} q_{c,0}(x - ct + x_1) \\ r_{c,0}(x - ct - x_1) \end{pmatrix} \tag{8}$$

where $0 \neq x_1 \in \mathbf{R}$. We will refer to $\mathbf{p}_{c,0}$ as a "piggybacking" solitary wave[4] as we can view the solution in the one component as riding on the back of the other—see Figure 1. The solutions $\mathbf{t}_{c,0}$ and $\mathbf{b}_{c,0}$ exist for $c \in I_1$ and $c \in I_2$ respectively, while the solutions $\mathbf{p}_{c,0}$ and $\mathbf{p}_{c,0,x_1}$ exist for $c \in I_0$.

[4]We have chosen to use the notation $\mathbf{t}$, $\mathbf{b}$ and $\mathbf{p}$ to reflect that $\mathbf{t}$ is non-zero in the *top* component, $\mathbf{b}$ is non-zero in the *bottom* component and $\mathbf{p}$ is the *piggybacking* solution.

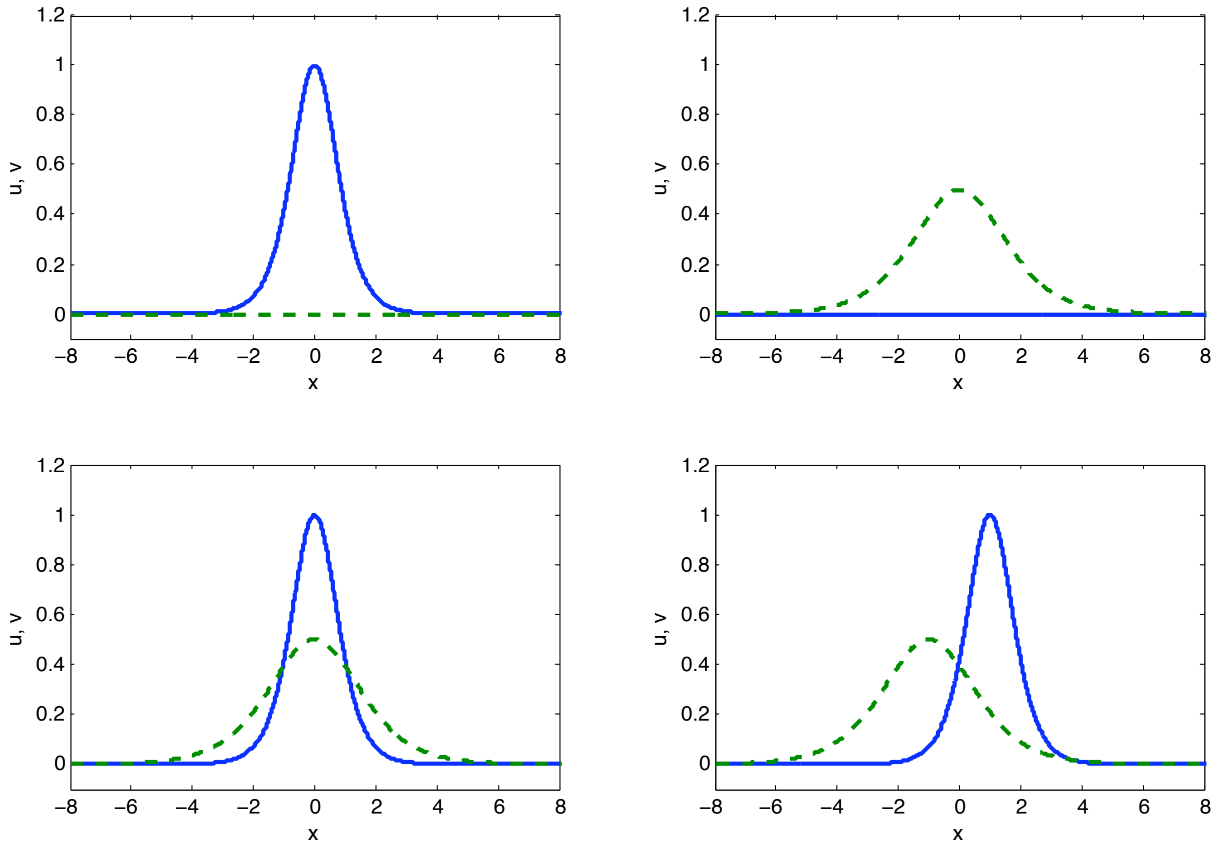

Figure 1: The four types of uncoupled solitary waves. Dashed lines represent the u component and solid lines the v component. The "piggybacking" solitary wave is in the bottom left.

When the coupling is "turned on" it is easy to show that solutions akin to (5)-(7) exist. This follows from a straight-forward application of the implicit function theorem, which we demonstrate in Section 3. It is less clear that the decoupled solitary waves described by (8) give rise to solutions in the coupled problem. In Section 4 we derive conditions which are necessary and sufficient to guarantee the existence of this type of solution.

More precisely, we have the following theorems on the existence of solitary waves:

Theorem 4. *For $c \in I_0$ there exist (unique) C^∞ maps from a neighborhood of zero into the subspace of even functions in $H^2 \times H^2$*

$$\begin{aligned} \epsilon &\longmapsto \mathbf{t}_{c,\epsilon}(y) \\ \epsilon &\longmapsto \mathbf{b}_{c,\epsilon}(y) \\ \epsilon &\longmapsto \mathbf{p}_{c,\epsilon}(y) \end{aligned}$$

such that $E'_\epsilon[\mathbf{t}_{c,\epsilon}] + cM'[\mathbf{t}_{c,\epsilon}] = 0$, $E'_\epsilon[\mathbf{b}_{c,\epsilon}] + cM'[\mathbf{b}_{c,\epsilon}] = 0$ and $E'_\epsilon[\mathbf{p}_{c,\epsilon}] + cM'[\mathbf{p}_{c,\epsilon}] = 0$. When $\epsilon = 0$ the functions $\mathbf{t}_{c,\epsilon}(y)$, $\mathbf{b}_{c,\epsilon}(y)$, $\mathbf{p}_{c,\epsilon}(y)$ are equal to those functions in (5)-(7). All functions decay at an exponential rate as $|x|$ goes to infinity.

Remark 5. *Since the maps in the above theorem are C^∞, notice that the size of the v component of $\mathbf{t}_{c,\epsilon}$ is $O(\epsilon)$—as is the u component of $\mathbf{b}_{c,\epsilon}$.*

Theorem 6. *Let*

$$\alpha(x_1) = \int q'_{c,0}(x + x_1) \left(K_3 r_{c,0}(x - x_1) + \partial_u H(q_{c,0}(x + x_1), r_{c,0}(x - x_1)) \right) dx. \tag{9}$$

If $\alpha(x_\star) = 0$ and $\alpha'(x_\star) \neq 0$ then there exist a C^∞ map from a neighborhood of zero into $H^2 \times H^2$

$$\epsilon \longmapsto \mathbf{p}_{c,\epsilon,x_\star}(y)$$

such that $E'_\epsilon[\mathbf{p}_{c,\epsilon,x_\star}] + cM'[\mathbf{p}_{c,\epsilon,x_\star}] = 0$. All functions decay at an exponential rate as $|x|$ goes to infinity. When $\epsilon = 0$ the function $\mathbf{p}_{c,\epsilon,x_\star}(y)$ is equal to that in (8). Note that c is restricted to the interval I_0. If $\alpha(x_1) \neq 0$ then there is no solution for small non-zero ϵ. Finally, for fixed $x_\star$, the solitary waves $\mathbf{p}_{c,\epsilon,x_\star}$ are unique up to translation.

Remark 7. *The function $\mathbf{q}_{c,0}$ is even and its derivative is odd. As a result the integrand of $\alpha(0)$ is odd, and therefore $\alpha(0) = 0$. That is to say, Theorem 4 is a corollary of Theorem 6. Nonetheless, there is a very simple proof of Theorem 4 which we also carry out.*

Remark 8. *The notation used for the piggybacking waves, $\mathbf{p}_{c,\epsilon,x_\star}$, should not be taken to mean that the "separation" between the waves remains $x_\star$ for $\epsilon \neq 0$. This value varies continuously with ϵ, but we leave the perturbed wave named after its unperturbed separation for simplicity.*

2.2 Stability

If we consider solitary waves of all four types with equal momentum, a routine calculation shows that it is either $\mathbf{t}_{c,\epsilon}$ or $\mathbf{b}_{c,\epsilon}$ which has the least energy (at least for the Gear-Grimshaw model). One can conclude that these are the constrained minimizers described in the previous works [1] and [4]. If a solitary wave solution $\mathbf{q}_{c,\epsilon}$ can be shown to be even a *local* minimizer of the energy under the constraint of fixed momentum, then the solution is (typically) stable. Equivalently, a solitary wave is stable provided the operator

$$L_\epsilon \equiv E''_\epsilon[\mathbf{q}_{c,\epsilon}] + cM''[\mathbf{q}_{c,\epsilon}]$$

is definite when restricted to set of functions with momentum equal to that of $\mathbf{q}_{c,\epsilon}$. This is more or less in analogy with finite dimensional hamiltonian problems, where equilibria are stable if the hessian of the energy at the fixed point is definite. Benjamin was the first to formalize this approach in [3] and the most general treatment is carried out in the works [11] and [12] by Grillakis, Shatah & Strauss. Therein, the conditions for determining the stability are reduced to the following simple criteria: if (a) $E''_\epsilon[\mathbf{q}_{c,\epsilon}] + cM''[\mathbf{q}_{c,\epsilon}]$ has at most one negative eigenvalue and (b) $\partial_c M[\mathbf{q}_{c,\epsilon}]$ is positive then $\mathbf{q}_{c,\epsilon}$ is orbitally stable. We remark that $L_\epsilon \mathbf{q}'_{c,\epsilon} = 0$ regardless of which type of solitary wave we are dealing with—a consequence of the translation invariance of the problem—and so L_ϵ always has a zero eigenvalue. Because we want the solitary waves in each component to be orbitally stable when viewed independently, we will make the following hypothesis:

Hypothesis 9. *$\partial_c \|q_{c,0}\|_{L^2}$ and $\partial_c \|r_{c,0}\|_{L^2}$ are positive.*

We shall show that the solitary waves $\mathbf{t}_{c,\epsilon}$ and $\mathbf{b}_{c,\epsilon}$ satisfy both conditions (a) and (b) and are therefore orbitally stable. On the other hand the solutions $\mathbf{p}_{c,0}$ and $\mathbf{p}_{c,0,x_1}$ to the uncoupled problem are necessarily unstable; a slight adjustment to the wave-speed in either the u or v component causes the waves to separate. From our point of view, even though in each component the resulting solution is a solitary wave, their speeds are different and so the solution is not a solitary wave *for the system*. An obvious question is whether or not the coupling can arrest this instability. As we shall see, these solutions violate the eigenvalue condition (a) above and so we will compute the spectrum of the linearization of the equation about these solutions to get a better understanding of their stability. Related to the stability of the piggybacking solitary waves is an interesting phenomenon known to exist in equations of this type called "leapfrogging". This is a behavior (observed by Liu, Kubota & Ko in [14], Gear & Grimshaw in [9] and investigated by Malomed in [15]) in which the solitary waves in u and v take turns "leading the way." That is, the solution oscillates about some common center of mass. Similar phenomena are known to occur in coupled nonlinear Schrödinger equations, see Malomed [15], and Goodman & Haberman [10]. In our analysis of the linear stability of the piggybacking solitary waves, we see that this behavior corresponds to an oscillatory *instability*.

If one linearizes (2) about a solitary wave $\mathbf{q}_{c,\epsilon}(x-ct)$ (which may be any of the solitary waves described above) in the moving reference frame $y = x - ct$ one arrives at the equation

$$\mathbf{w}_t = \partial_y \left(E''_\epsilon[\mathbf{q}_{c,\epsilon}] + cM''[\mathbf{q}_{c,\epsilon}] \right) \mathbf{w} = \partial_y L_\epsilon \mathbf{w}.$$

To understand the linearized operator above it is necessary to first understand the linearization of a single generalized KdV equation. Pego & Weinstein carried out an exhaustive analysis of such equations in their seminal paper [17]. We briefly recall their results here. If $q(x-ct)$ is a solitary wave solution[5] of $u_t + (u_{xx} + u^p)_x = 0$, $2 \leq p < 5$, the linearization of the equation about this solution (in the frame of reference $y = x - ct$) is

$$w_t = \partial_y \left(cw - w'' - pq^{p-1} w \right) \equiv \partial_y L w.$$

It is elementary to check that $Lq' = 0$. In fact, q' spans the kernel and L has Fredholm index zero. Moreover, since L is a second order differential operator and $q'(y)$ crosses the y axis only one time we can use Sturm-Liouville theory to conclude that L has one (and only one) negative eigenvalue. Clearly $\partial_y L q' = 0$ as well. Moreover, we have $\partial_y L \partial_c q = -q'$. Thus the zero eigenvalue of $\partial_y L$ is (at least) algebraically double. In [17] it is shown that if $p = 2,\ 3$, or 4 then there are no additional eigenvalues of $\partial_y L$ but that as p goes through 5 an additional unstable eigenvalue is produced.[6]

Since the function q decays exponentially at spatial infinity, L is a compact perturbation of $c - \partial_x^2$ and their essential spectra (as operators on L^2) coincide. The essential spectrum of

[5] q is even, positive, decays exponentially and has only one local maximum.

[6] In the case where $2 \leq p < 5$ it has been established that the solitary waves are in fact asymptotically stable—see Pego & Weinstein [18] and Martel & Merle [16]

$c-\partial_x^2$ can be computed *via* the Fourier transform and is the set of all real numbers greater than c; since c is strictly positive this lies in the right half plane. The same reasoning leads to the conclusion that the essential spectrum of $\partial_y L$ is the imaginary axis, thus the eigenvalue at zero is embedded.

In this light, we make the following hypotheses:

Hypothesis 10. *The kernels of $(c - K_1 - F_1''(q_{c,0}))$ and $(c - K_2 - F_2''(r_{c,0}))$ consist solely of $q'_{c,0}$ and $r'_{c,0}$ respectively and these operators have Fredholm index zero. Each has one (and only one) additional negative eigenvalue. The essential spectra of these operators (as operators on L^2) are, respectively, all reals greater than c and all reals greater than $c - c_1$.*

Remark 11. *Hypothesis 10 is in fact automatically satisfied for the operators K_i we are considering in this article. Nonetheless, we prefer to make this an hypothesis so that our results are adaptable to more general choices for these operators.*

Hypothesis 12. *The spectrum of $\partial_y\left(c - K_1 - F_1''(q_{c,0})\right)$ (as an operator on L^2) is made up of the essential spectrum, which lies along the imaginary axis, and a geometrically single, algebraically double eigenvalue at zero. The same statement is true for $\partial_y\left(c - K_2 - F_2''(r_{c,0})\right)$.*

Remark 13. *Hypothesis 12 is made in part to exclude situations where there is a triple eigenvalue at the origin—a situation which arises for "critical nonlinearities" [17].*

These hypotheses determine the spectrum of L_0 and $\partial_y L_0$. First we discuss L_0. For $\mathbf{t}_{c,0}$,

$$L_0 = \begin{pmatrix} c - \partial_x^2 - F_1''(q_{c,0}) & 0 \\ 0 & c - c_1 - c_2\partial_x^2 \end{pmatrix}$$

and the spectrum will be the zero eigenvalue (with eigenfunction $(q'_{c,0}, 0)^t$)), the negative eigenvalue and the essential spectrum which, in this case, is all reals greater than the minimum of $c - c_1$ and c. If c is in I_0 (which is to say if $c > c_1$), then we see that the essential spectrum is in the right half plane. Thus we can can conclude that L_0 is of Fredholm index zero. (If c is not in I_0, then the zero eigenvalue would be embedded in the essential spectrum.) Since $L_\epsilon \mathbf{t}'_{c,\epsilon} = 0$ for all ϵ, we see that under perturbations the zero eigenvalue does not move. Thus the spectrum of L_ϵ is qualitatively identical to that of L_0, and we conclude that $\mathbf{t}_{c,\epsilon}$ is orbitally stable by appealing to the results cited above. Similar arguments show that $\mathbf{b}_{c,\epsilon}$ is also stable.

On the other hand, if we are dealing with $\mathbf{p}_{c,\epsilon}$ or $\mathbf{p}_{c,\epsilon,x_\star}$ then

$$L_0 = \begin{pmatrix} c - \partial_x^2 - F_1''(q_{c,0}) & 0 \\ 0 & c - c_1 - c_2\partial_x^2 - F_2''(r_{c,0}) \end{pmatrix}.$$

The essential spectrum remains as for $\mathbf{t}_{c,\epsilon}$, but now there is a geometrically double zero eigenvalue (with eigenfunctions $\mathbf{p}'_{c,0} = (q'_{c,0}, r_{c,0})^t$ and $(q'_{c,0}, -r'_{c,0})^t$) and two negative eigenvalues, making an application of the abstract theory of Grillakis, Shatah & Strauss unavailable. Notice that the zero eigenfunctions can be interpreted as being generated by (in the first case) the translation invariance in the problem and (in the second case) by the separation invariance. As we have already remarked, the eigenvalue generated by the translation invariance will stay put under perturbations (since $L_\epsilon \mathbf{p}'_{c,\epsilon} = 0$). But the "separation eigenvalue" will split from zero. We compute this splitting in Section 5 and interpret the results.

Since L_ϵ is self-adjoint, the new eigenvalue will be either positive or negative. It is not immediately clear how this splitting effects the stability of the wave. We find more information about the stability of piggybacking (or asymmetric) solitary waves by examining the spectrum the linearization $\partial_y L_\epsilon$. The spectrum of $\partial_y L_0$ is the essential spectrum (which is the imaginary axis) and the algebraically quadruple, geometrically double zero eigenvalue. (Geometrically double due to the translation invariance and the separation invariance.) $\partial_y L_\epsilon \mathbf{p}'_{c,\epsilon} = 0$ for all ϵ and taking a derivative of $E'[\mathbf{p}_{c,\epsilon}] + cM'[\mathbf{p}_{c,\epsilon}] = 0$ with respect to c leads to $\partial_y L_\epsilon \partial_c \mathbf{p}_{c,\epsilon} = -\mathbf{p}'_{c,\epsilon}$.

It turns out that under perturbations the zero eigenvalue remains at least algebraically double. But the separation symmetry is broken as ϵ is changed from zero and two new eigenvalues split from the origin. Note that since the origin is part of the essential spectrum, we compute the spectrum of the operator in an exponentially weighted function space as opposed to $L^2 \times L^2$. The essential spectrum of the operator in such a space does not pass through the origin, thus simplifying the perturbation analysis.

The system is hamiltonian and so one's first instinct is to expect that the eigenvalues split along the real *or* the imaginary axis. In the former case, the piggybacking solitary wave is obviously unstable. In the latter, we would expect this solitary wave to be neutrally stable, as would occur in a finite dimensional problem. This does not occur! Since ∂_x is not invertible the intuition we have from hamiltonian ordinary differential equations fails. What we discover is that for $\epsilon \neq 0$ there is always at least one eigenvalue which has positive real part.

Specifically, the splitting of the eigenvalues is linked to the quantity

$$\kappa = \frac{1}{4} \left\langle \begin{pmatrix} q'_{c,0} \\ -r'_{c,0} \end{pmatrix}, L_{0,1} \begin{pmatrix} q'_{c,0} \\ -r'_{c,0} \end{pmatrix} \right\rangle, \tag{10}$$

where

$$L_{0,1} = \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \left(E''_\epsilon[\mathbf{p}_{c,\epsilon,x_\star}] + cM''[\mathbf{p}_{c,\epsilon,x_\star}] \right).$$

κ is the leading order change (in ϵ) of the energy if the solitary wave $(q_{c,\epsilon}, r_{c,\epsilon})^t$ is perturbed in the "direction" $(q'_{c,\epsilon}, -r'_{c,\epsilon})^t$. This direction corresponds to an attempt to separate the maximum of the u and v components of the solitary wave away from their common center.

We have the following theorems regarding the spectra of L_ϵ and $\partial_y L_\epsilon$. Note that $L^2_a = \left\{ u(y) | e^{ay} u(y) \in L^2 \right\}$.

Theorem 14. *For ϵ sufficiently close to zero, the spectrum of $E''_\epsilon[\mathbf{p}_{c,\epsilon,x_\star}] + cM''[\mathbf{p}_{c,\epsilon,x_\star}]$ in $L^2 \times L^2$ consists of (a) the essential spectrum, which lies in the right half plane and is purely real (b) two negative real eigenvalues (c) a zero eigenvalue and (d) one additional eigenvalue located at $\lambda = -C\epsilon\kappa + O(\epsilon^2)$ where C is a positive non-zero constant which is independent of ϵ.*

Theorem 15. *There exists $a_0 > 0$ such that the following is true for all $a \in (0, a_0)$. For ϵ sufficiently close to zero, the spectrum of $\partial_y(E''_\epsilon[\mathbf{p}_{c,\epsilon,x_\star}] + cM''[\mathbf{p}_{c,\epsilon,x_\star}])$ in $L^2_a \times L^2_a$ consists of (a) the essential spectrum, which lies in the left half plane[7] (b) an algebraically double, geometrically single zero eigenvalue and (c) two additional eigenvalues located at*

$$\lambda_\pm = C_2\epsilon\kappa \pm \sqrt{-C_3\epsilon\kappa + O(\epsilon^2)}$$

where C_2 and C_3 are positive constants independent of ϵ. Note that if $\epsilon\kappa < 0$ then there is a positive eigenvalue of $O(\sqrt{|\epsilon|})$. If $\epsilon\kappa > 0$ then there are two complex eigenvalues with positive real parts of $O(\epsilon)$. That is to say, the solitary waves $\mathbf{p}_{c,\epsilon,x_\star}$ are always linearly unstable in $L^2_a \times L^2_a$.

This result is marred somewhat by the fact that it relies on the use of the exponentially weighted spaces. We have the following Corollary of Theorem 15 which remedies this issue.

Corollary 16. *For all ϵ sufficient close to zero but not equal to zero, the spectrum of $\partial_y(E''_\epsilon[\mathbf{p}_{c,\epsilon,x_\star}] + cM''[\mathbf{p}_{c,\epsilon,x_\star}])$ in $L^2 \times L^2$ contains at least one eigenvalue with positive real part. More specifically, if $\epsilon\kappa < 0$ then there is a single real positive eigenvalue of $O(\sqrt{|\epsilon|})$. If $\epsilon\kappa > 0$ then there is a complex conjugate pair with positive real parts of $O(\epsilon)$. That is to say, the solitary waves $\mathbf{p}_{c,\epsilon,x_\star}$ are always linearly unstable in $L^2 \times L^2$.*

[7] see (18) for an explicit characterization of the essential spectrum

Note that if the eigenvalues are complex, then the instability is an oscillatory one. This is the origin of the leapfrogging behavior described earlier. We remark that the formal analysis in [15] shows that the frequency of the leapfrogging oscillation is $O(\sqrt{|\epsilon|})$, which coincides with our result.

3 Symmetric solitary waves

In this section we will prove Theorem 4. Specifically we will prove the existence of the piggybacking solution $\mathbf{p}_{c,\epsilon}$. The proofs for the existence of $\mathbf{t}_{c,\epsilon}$ and $\mathbf{b}_{c,\epsilon}$ differ only in minor details.

We look for a solution to (2) of the form

$$\mathbf{u}(x-ct) = \mathbf{p}_{c,\epsilon}(x-ct) \equiv \begin{pmatrix} q_{c,\epsilon}(x-ct) \\ r_{c,\epsilon}(x-ct) \end{pmatrix}.$$

The profile function $\mathbf{p}_{c,\epsilon}(y)$ must satisfy

$$E'_\epsilon[\mathbf{p}_{c,\epsilon}] + cM'[\mathbf{p}_{c,\epsilon}] = 0.$$

Hypothesis 2 tells us that

$$E'_0[\mathbf{p}_{c,0}] + cM'[\mathbf{p}_{c,0}] = 0.$$

Moreover, Hypothesis 10 implies that the linearization of $E'_0 + cM'$ at $\mathbf{p}_{c,0}$, $L_0 = E''_0[\mathbf{p}_{c,0}] + cM''[\mathbf{p}_{c,0}]$, has kernel given by

$$\ker L_0 = \operatorname{span}\left\{ \mathbf{v}_1 \equiv \frac{1}{\|q'_{c,0}\|_{L^2}} \begin{pmatrix} q'_{c,0}(y) \\ 0 \end{pmatrix}, \mathbf{v}_2 \equiv \frac{1}{\|r'_{c,0}\|_{L^2}} \begin{pmatrix} 0 \\ r'_{c,0}(y) \end{pmatrix} \right\}.$$

Note that $\mathbf{v}_1$ and $\mathbf{v}_2$ form an orthonormal basis for the kernel.

Since the kernel of L_0 is not trivial, it seems our attempt to use the implicit function theorem will not succeed. But consider the Hilbert space

$$\mathbf{H}^s_{\text{even}} = \{\mathbf{u} \in H^s(\mathbf{R}) \times H^s(\mathbf{R}) \text{ and } \mathbf{u}(y) = \mathbf{u}(-y)\}.$$

By hypothesis, we know that $\mathbf{p}_{c,0}$ is in this space. (As are $\mathbf{t}_{c,0}$, $\mathbf{b}_{c,0}$. If $x_1 \neq 0$, then $\mathbf{p}_{c,0,x_1}$ is not.) In addition, $E'_\epsilon + cM'$ maps $\mathbf{H}^2_{\text{even}}$ into $\mathbf{H}^0_{\text{even}}$. Since $q_{c,0}$ and $r_{c,0}$ are hypothesized to be even functions, their derivatives are odd. Thus $\ker L_0 \cap \mathbf{H}^0_{\text{even}} = \{0\}$. The space $L^2 \times L^2$ is the direct sum of the subspaces of even and odd functions and L_0 respects this decomposition. Therefore, since L_0 is Fredholm index zero, the restriction of L_0 to $\mathbf{H}^0_{\text{even}}$ is an index zero map as well and hence has a closed range. So we can apply the implicit function theorem to discover the existence of a (unique) C^∞ map from a neighborhood of $\epsilon = 0$ into $\mathbf{H}^2_{\text{even}}$

$$\epsilon \longmapsto \mathbf{p}_{c,\epsilon} = \begin{pmatrix} q_{c,\epsilon} \\ r_{c,\epsilon} \end{pmatrix}$$

such that $E'_\epsilon[\mathbf{p}_{c,\epsilon}] + cM'[\mathbf{p}_{c,\epsilon}] = 0$.

We need only to verify that $\mathbf{p}_{c,\epsilon}$ decays exponentially as $|x|$ goes to infinity. Since the function $\mathbf{p}_{c,\epsilon}$ is in $H^2 \times H^2$, it is in fact continuous by the standard Sobolev embedding theorem. As such, $\mathbf{p}_{c,\epsilon}$ is a classical solution to (3) and behaves for large x like a decaying solution to the linear problem

$$\begin{aligned} -cq + K_1 q + \epsilon K_3 r &= 0 \\ -cr + K_2 r + \epsilon K_3 q &= 0. \end{aligned}$$

We have assumed that zero is an hyperbolic fixed point for the above system when $\epsilon = 0$ and as such it remains hyperbolic under the perturbation. Therefore decaying solutions to the linear problem go to zero exponentially, and so does $\mathbf{p}_{c,\epsilon}$. This concludes the proof of Theorem 4.

4 Asymmetric solitary waves

Since the linearization of $E'_\epsilon + cM'$ is not invertible for general functions in $H^2 \times H^2$, we cannot use the implicit function theorem argument above to establish the existence of asymmetric solitary wave solutions to (2) which are analogous to $\mathbf{p}_{c,0,x_1}$ in the uncoupled problem. Instead we will apply a Liapunov-Schmidt analysis.

Define the mapping mapping

$$\Phi(\mathbf{u};\epsilon) = E'_\epsilon[\mathbf{u}] + cM'[\mathbf{u}].$$

We have $\Phi(\mathbf{p}_{c,0,x_1};0) = 0$. Let L_0 be the derivative of Φ with respect to $\mathbf{u}$ and evaluated at $\mathbf{u} = \mathbf{p}_{c,0,x_1}$ and $\epsilon = 0$. Specifically

$$L_0 = \begin{pmatrix} c - K_1 - F_1''(q_{c,0}(y+x_1)) & 0 \\ 0 & c - K_2 - F_2''(r_{c,0}(y-x_1)) \end{pmatrix}. \tag{11}$$

Hypothesis 10 tells us that the kernel of L_0 is

$$\mathcal{E}_0 = \ker L_0 = \operatorname{span}\left\{ \mathbf{v}_1 \equiv \frac{1}{\|q'_{c,0}\|_{L^2}} \begin{pmatrix} q'_{c,0}(y+x_1) \\ 0 \end{pmatrix}, \mathbf{v}_2 \equiv \frac{1}{\|r'_{c,0}\|_{L^2}} \begin{pmatrix} 0 \\ r'_{c,0}(y-x_1) \end{pmatrix} \right\}.$$

L_0 is self-adjoint and by Hypothesis 10 has Fredholm index equal to zero. Therefore its range, $\mathcal{R}$, is simply ${\mathcal{E}_0}^\perp$.[8]

Let P be the orthogonal projection of $L^2 \times L^2$ onto $\mathcal{E}_0$

$$P\ \mathbf{f} = \langle \mathbf{f}, \mathbf{v}_1 \rangle \mathbf{v}_1 + \langle \mathbf{f}, \mathbf{v}_2 \rangle \mathbf{v}_2. \tag{12}$$

Therefore, $\Phi(\mathbf{u};\epsilon) = 0$ if and only if

$$\begin{aligned} P\ \Phi(\mathbf{u};\epsilon) &= 0, \text{ and} \\ (1-P)\ \Phi(\mathbf{u};\epsilon) &= 0 \end{aligned}$$

simultaneously. $1-P$ is the projection onto $\mathcal{R}$.

Let $\mathbf{u} = \mathbf{p}_{c,0,x_1} + \mathbf{d}$. Since $\mathbf{d} = P\ \mathbf{d} + (1-P)\ \mathbf{d}$, the equation $(1-P)\ \Phi = 0$ can be rewritten as

$$\Phi_{\mathcal{R}}(\mathbf{d}_0, \mathbf{d}_1;\epsilon) \equiv (1-P)\ \Phi(\mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{d}_1;\epsilon) = 0 \tag{13}$$

where $\mathbf{d}_0 = P\ \mathbf{d}$ and $\mathbf{d}_1 = (1-P)\ \mathbf{d}$. $\Phi_{\mathcal{R}}$ is a map from $\mathcal{E}_0 \times \mathcal{R} \times \mathbf{R}$ into $\mathcal{R}$. By construction, the derivative of $\Phi_{\mathcal{R}}$ with respect to $\mathbf{d}_1$ has a trivial kernel. So, by the implicit function theorem we conclude that there is a map

$$\mathbf{h}(\mathbf{d}_0;\epsilon)$$

from $\mathcal{E}_0 \times \mathbf{R}$ into $\mathcal{R}$ such that

$$\Phi_{\mathcal{R}}\left(\mathbf{d}_0, \mathbf{h}(\mathbf{d}_0,\epsilon);\epsilon\right) = 0.$$

(Note that $\mathbf{h}(0;0) = 0$.)

Now we must solve the equation $P\ \Phi = 0$. More specifically, solutions to $\Phi = 0$ are in one to one correspondence with solutions of

$$\Phi_{\mathcal{E}_0}(\mathbf{d}_0;\epsilon) \equiv P\ \Phi(\mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{h}(\mathbf{d}_0;\epsilon);\epsilon) = 0.$$

Recall that Φ is the gradient of $E_\epsilon + cM$. It happens that $\Phi_{\mathcal{E}_0}$ is also a gradient of a functional on $\mathcal{E}_0$. We denote the inner product on $\mathcal{E}_0$ by $\langle \cdot, \cdot \rangle_{\mathcal{E}_0}$ and this is simply $\langle P\cdot, P\cdot \rangle$ restricted to $\mathcal{E}_0$. Specifically we have the following Lemma:

[8] Note that perpendicular spaces are found with respect to the $L^2 \times L^2$ inner product, even though $\mathcal{E}_0$ is a set in $H^2 \times H^2$.

Lemma 17. $\Phi_{\mathcal{E}_0}(\mathbf{d}_0;\epsilon) = \nabla_0 V(\mathbf{d}_0;\epsilon)$ *where*

$$V(\mathbf{d}_0;\epsilon) = (E_\epsilon + cM)[\mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{h}(\mathbf{d}_0;\epsilon)].$$

Here, ∇_0 *is the gradient on* $\mathcal{E}_0$.

Proof. We will suppress ϵ dependence for convenience. First notice that the chain rule implies

$$\nabla_0 V(\mathbf{d}_0) = (I_{\mathcal{E}_0} + D\mathbf{h})^\dagger \Phi(\mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{h}(\mathbf{d}_0))$$

where $I_{\mathcal{E}_0}$ is the identity map on $\mathcal{E}_0$, $D\mathbf{h}$ (the derivative of $\mathbf{h}$ with respect to $\mathbf{d}_0$) is a linear map from $\mathcal{E}_0$ into $\mathcal{R}$ and a superscript "†" denotes the adjoint. $(D\mathbf{h})^\dagger$ is linear from $\mathcal{R}$ into $\mathcal{E}_0$ and $I_{\mathcal{E}_0}^\dagger$ is P. Therefore

$$\begin{aligned} \nabla_0 V(\mathbf{d}_0) =& P\ \Phi(\mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{h}(\mathbf{d}_0)) + (D\mathbf{h})^\dagger \Phi(\mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{h}(\mathbf{d}_0)) \\ =& \Phi_{\mathcal{E}_0}(\mathbf{d}_0) + (D\mathbf{h})^\dagger \Phi(\mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{h}(\mathbf{d}_0)) \\ =& \Phi_{\mathcal{E}_0}(\mathbf{d}_0) + (D\mathbf{h})^\dagger P \Phi(\mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{h}(\mathbf{d}_0)). \end{aligned} \tag{14}$$

In the last line we have used that fact that, by construction, $(1-P)\ \Phi(\mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{h}(\mathbf{d}_0)) = 0$. Furthermore, $P\ D\mathbf{h} = 0$ since $D\mathbf{h}$ maps into $\mathcal{R}$. Since P is an orthogonal projection we find that $0 = (P\ D\mathbf{h})^\dagger = (D\mathbf{h})^\dagger P$. Thus the second term in the final line of (14) is zero and the proof of the Lemma is complete. □

This Lemma implies that the critical points of $E_\epsilon + cM$ are in one to one correspondence with those of V. Moreover, when $\epsilon = 0$ we can translate the components of a critical point of $E_0 + cM$ independently without affecting that point's criticality. This implies that $\nabla_0 V(\mathbf{d}_0; 0) = 0$ for all $\mathbf{d}_0$. Without loss of generality we will assume $V(\mathbf{d}_0; 0) = 0$.

Due to the translation invariance of the problem, for any function $\mathbf{u}$ we have

$$\langle \Phi(\mathbf{u};\epsilon), \partial_x \mathbf{u} \rangle = 0.$$

This invariance in the original functional is carried over into the reduced functional V in the following way. Let $\mathbf{p}(\mathbf{d}_0;\epsilon) = \mathbf{p}_{c,0,x_1} + \mathbf{d}_0 + \mathbf{h}(\mathbf{d}_0;\epsilon)$. Then

$$\begin{aligned} 0 =& \langle \Phi(\mathbf{p};\epsilon), \partial_x \mathbf{p} \rangle \\ =& \langle (1 - P + P)\Phi(\mathbf{p};\epsilon), \partial_x \mathbf{p} \rangle \\ =& \langle P\ \Phi(\mathbf{p};\epsilon), \partial_x \mathbf{p} \rangle \\ =& \langle P\ \Phi(\mathbf{p};\epsilon), P\partial_x \mathbf{p} \rangle \\ =& \langle \nabla_0 V(\mathbf{d}_0;\epsilon), \mathbf{f}(\mathbf{d}_0;\epsilon) \rangle_{\mathcal{E}_0} \end{aligned}$$

where $\mathbf{f}(\mathbf{d}_0;\epsilon) = P\partial_x \mathbf{p}(\mathbf{d}_0;\epsilon)$. Notice that we can view $\mathbf{f}$ as a vector field on $\mathcal{E}_0$ (which is to say, $\mathbf{R}^2$) with the property that it is everywhere orthogonal to $\nabla_0 V$. Thus, if we can find $\mathbf{d}_0$ such that $\nabla V(\mathbf{d}_0;\epsilon)$ is orthogonal to $\mathbf{g}(\mathbf{d}_0)$, where $\mathbf{g}(\mathbf{d}_0)$ is any vector field transverse to $\mathbf{f}(\mathbf{d}_0;\epsilon)$, then $\mathbf{d}_0$ is a critical point of V. We could take $\mathbf{g}(\mathbf{d}_0) = \mathbf{f}^\perp(\mathbf{d}_0;\epsilon)$, but we will pick an alternate vector field which simplifies some calculations.

Moreover V is constant along solutions to the ordinary differential equation $\frac{d}{d\tau}\mathbf{d} = \mathbf{f}(\mathbf{d};\epsilon)$. Thus, if $\mathbf{d}_0$ is a critical point then so are all points along its orbit under this flow. This corresponds to the fact that translations of the solitary waves in the full problem are also solitary waves. Since $\mathcal{E}_0$ is two-dimensional, this fact allows us to reduce the dimension of the problem by one.

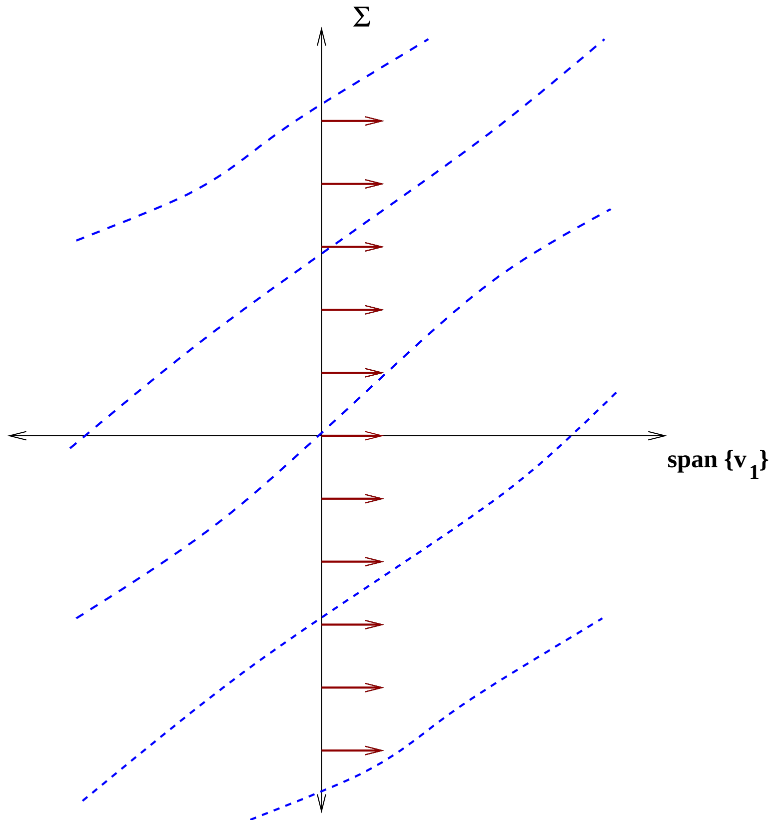

Figure 2: Sketch of $\mathcal{E}_0$, Σ and the reduced problem. V is constant along dashed lines. The vector field $\mathbf{g}$ is depicted as arrows along Σ.

We do this as follows. Let Σ be the one dimensional subspace of $\mathcal{E}_0$ given by the span of $\mathbf{v}_2$. Σ is transverse to $f(\mathbf{d}_0;0)$ near the origin. To wit,

$$\begin{aligned}\mathbf{f}(0;0) =& P\partial_x \mathbf{p}_{c,0,x_1}\\ =&\|q'_{c,0}\|\mathbf{v}_1 + \|r'_{c,0}\|\mathbf{v}_2.\end{aligned}$$

For this reason, we also choose the transverse vector field $\mathbf{g}$ to be simply the constant $\mathbf{v}_1$. (See Figure 2.) Let $\eta(\sigma;\epsilon) = \langle \nabla_0 V(\sigma\mathbf{v}_2;\epsilon), \mathbf{v}_1\rangle_{\mathcal{E}_0}$. If we find σ such that $\eta(\sigma;\epsilon) = 0$ then we are done. Since $V(\mathbf{d}_0;0) = 0$, $\eta(\sigma;0) = 0$ for all σ. Thus we can write $\eta(\sigma;\epsilon) = \epsilon\eta_1(\sigma;\epsilon)$ where

$$\eta_1(\sigma;\epsilon) = \frac{1}{\epsilon}\int_0^\epsilon \partial_\epsilon \eta(\sigma;\gamma)d\gamma.$$

For $\epsilon \neq 0$, we can solve $\eta(\sigma;\epsilon) = 0$ only if $\eta_1(\sigma;\epsilon) = 0$. Thus we need $\eta_1(0;0) = 0$. The fundamental theorem of calculus tells us that this is equivalent to requiring $\partial_\epsilon\eta(0;0) = 0$. If $\partial_\sigma\eta_1(0;0) \neq 0$ (or equivalently $\partial_\sigma\partial_\epsilon\eta(0;0) \neq 0$) then the implicit function theorem asserts the existence of a function $s(\epsilon)$ such that $\eta_1(s(\epsilon),\epsilon) = 0$, and therefore also the existence of solitary waves in the original problem.

The conditions that guarantee existence, $\partial_\epsilon\eta(0;0) = 0$ and $\partial_\epsilon\partial_\sigma\eta(0;0) \neq 0$ are equivalent to the conditions $\alpha(x_\star) = 0$ and $\alpha'(x_\star) \neq 0$ given in the statement of Theorem 4. Specifically we have

$$\begin{aligned}\partial_\epsilon \eta(\sigma;0) =&\partial_\epsilon|_{\epsilon=0}\langle \nabla_0 V(\sigma;\epsilon), \mathbf{v}_1\rangle_{\mathcal{E}_0}\\ =&\partial_\epsilon|_{\epsilon=0}\langle P\Phi(\mathbf{p}_{c,0,x_1}+\sigma\mathbf{v}_2+\mathbf{h}(\sigma\mathbf{v}_2;\epsilon);\epsilon),\mathbf{v}_1\rangle\\ =&\langle P\partial_\epsilon\Phi(\mathbf{p}_{c,0,x_1}+\sigma\mathbf{v}_2+\mathbf{h}(\sigma\mathbf{v}_2;0);0),\mathbf{v}_1\rangle\\ &+\langle PD\Phi(\mathbf{p}_{c,0,x_1}+\sigma\mathbf{v}_2+\mathbf{h}(\sigma\mathbf{v}_2;0);0)\partial_\epsilon\mathbf{h}(\sigma\mathbf{v}_2;0),\mathbf{v}_1\rangle\end{aligned}$$

If $\sigma = 0$, then $PD\Phi(\mathbf{p}_{c,0,x_1};0) = PL_0 = 0$. Thus we have

$$\begin{aligned}\partial_\epsilon\eta(0;0) =&\langle\partial_\epsilon\Phi(\mathbf{p}_{c,0,x_1};0),\mathbf{v}_1\rangle\\ =&\frac{1}{\|q'_{c,0}\|}\alpha(x_1)\end{aligned}$$

where we have used the definitions of Φ and $\mathbf{v}_1$ to make the last step.

That $\partial_\sigma\partial_\epsilon\eta(0;0) \neq 0$ is equivalent to $\alpha'(x_1) \neq 0$ is not immediately evident. We have

$$\begin{aligned}\partial_\sigma\partial_\epsilon\eta(0;0) =&\partial_\sigma|_{\sigma=0}\langle P\partial_\epsilon\Phi(\mathbf{p}_{c,0,x_1}+\sigma\mathbf{v}_2+\mathbf{h}(\sigma\mathbf{v}_2;0);0),\mathbf{v}_1\rangle\\ &+\partial_\sigma|_{\sigma=0}\langle PD\Phi(\mathbf{p}_{c,0,x_1}+\sigma\mathbf{v}_2+\mathbf{h}(\sigma\mathbf{v}_2;0);0)\partial_\epsilon\mathbf{h}(\sigma\mathbf{v}_2;0),\mathbf{v}_1\rangle\\ =&\langle P\partial_\epsilon D\Phi(\mathbf{p}_{c,0,x_1};0)(\mathbf{v}_2+D\mathbf{h}(0;0)\mathbf{v}_2),\mathbf{v}_1\rangle\\ &+\langle PD\Phi(\mathbf{p}_{c,0,x_1};0)\partial_\epsilon D\mathbf{h}(0;0),\mathbf{v}_1\rangle\\ &+\langle PD^2\Phi(\mathbf{p}_{c,0,x_1};0)[\mathbf{v}_2+D\mathbf{h}(0;0)\mathbf{v}_2,\partial_\epsilon\mathbf{h}(0;0)],\mathbf{v}_1\rangle\end{aligned} \tag{15}$$

The following computation shows that $D\mathbf{h}(0;0) = 0$. By construction,

$$(1-P)\Phi(\mathbf{p}_{c,0,x_1}+\mathbf{d}_0+\mathbf{h}(\mathbf{d}_0;\epsilon);\epsilon) = 0.$$

We view the left hand side of this equation as a map from $\mathcal{E}_0\times\mathbf{R}$ into $\mathcal{R}$. Differentiating with respect to $\mathbf{d}_0$ and evaluating at $\mathbf{d}_0 = 0$ and $\epsilon = 0$ yields

$$(1-P)D\Phi(\mathbf{p}_{c,0,x_1};0)\left(I_{\mathcal{E}_0}+D\mathbf{h}(0;0)\right) = 0.$$

Since $D\Phi(\mathbf{p}_{c,0,x_1};0) = L_0$ is zero on $\mathcal{E}_0$, we have $D\Phi(\mathbf{p}_{c,0,x_1};0)I_{\mathcal{E}_0} = 0$. Therefore

$$(1-P)L_0D\mathbf{h}(0;0) = L_0D\mathbf{h}(0;0) = 0.$$

Moreover, $D\mathbf{h}(0;0)$ is a linear map from $\mathcal{E}_0$ into $\mathcal{R}$. The map L_0 is obviously invertible on its range, and so we conclude that $D\mathbf{h}(0;0) = 0$.

We use this result, along with the fact that $PL_0 = 0$, to simplify (15) to

$$\begin{aligned}\partial_\sigma\partial_\epsilon\eta(0;0) =&\langle P\partial_\epsilon D\Phi(\mathbf{p}_{c,0,x_1};0)\mathbf{v}_2,\mathbf{v}_1\rangle\\ &+\langle PD^2\Phi(\mathbf{p}_{c,0,x_1};0)[\mathbf{v}_2,\partial_\epsilon\mathbf{h}(0;0)],\mathbf{v}_1\rangle.\end{aligned}$$

It happens that the second line above is also identically equal to zero. A direct calculation shows that

$$\begin{aligned}&D^2\Phi(\mathbf{p}_{c,0,x_1};0)[\begin{pmatrix} f_1(x)\\ f_2(x)\end{pmatrix},\mathbf{h}]\\ =&\begin{pmatrix} -F_1'''(q_{c,0}(x+x_1))-q'_{c,0}(x+x_1)f_1(x) & 0\\ 0 & -F_2'''(r_{c,0}(x-x_1))r'_{c,0}(x-x_1)f_2(x)\end{pmatrix}\mathbf{h}.\end{aligned}$$

The function $\mathbf{v}_2$ is zero in the top component and thus $PD^2\Phi(\mathbf{p}_{c,0,x_1};0)[\mathbf{v}_2,\partial_\epsilon\mathbf{h}(0;0)]$ is a function which is zero in the top component. The inner product of this function with $\mathbf{v}_1$, which is zero in the bottom component, is clearly zero. Therefore

$$\partial_\sigma\partial_\epsilon\eta(0;0) =\langle P\partial_\epsilon D\Phi(\mathbf{p}_{c,0,x_1};0)\mathbf{v}_2,\mathbf{v}_1\rangle.$$

Now we compute $\alpha'(x_1)$.

$$\frac{1}{\|q'_{c,0}\|}\alpha'(x_1) = \frac{d}{dx_1}\langle \partial_\epsilon \Phi(\mathbf{p}_{c,0,x_1};0), \mathbf{v}_1\rangle$$
$$= \langle D\partial_\epsilon \Phi(\mathbf{p}_{c,0,x_1};0)\partial_{x_1}\mathbf{p}_{c,0,x_1}, \mathbf{v}_1\rangle + \langle \partial_\epsilon \Phi(\mathbf{p}_{c,0,x_1};0), \partial_{x_1}\mathbf{v}_1\rangle.$$

Noting that $\partial_{x_1}\mathbf{v}_1 = \partial_x \mathbf{v}_1$, $\partial_{x_1}\mathbf{p}_{c,0,x_1} = \|q'_{c,0}\|\mathbf{v}_1 - \|r'_{c,0}\|\mathbf{v}_2$ and $\partial_x \mathbf{p}_{c,0,x_1} = \|q'_{c,0}\|\mathbf{v}_1 + \|r'_{c,0}\|\mathbf{v}_2$, we integrate by parts in the second term in the above equation to find that

$$\frac{1}{\|q'_{c,0}\|}\alpha'(x_1) = -2\|r'_{c,0}\|\langle D\partial_\epsilon \Phi(\mathbf{p}_{c,0,x_1};0)\mathbf{v}_2, \mathbf{v}_1\rangle.$$

This last expression implies that $\alpha'(x_1)$ is a constant multiple of $\partial_\sigma \partial_\eta \eta(0;0)$ and we are done.

5 The spectrum of $E''_\epsilon + cM''$

As noted in Section 2, the spectrum of $L_\epsilon = E''_\epsilon[\mathbf{p}_{c,\epsilon,x_\star}] + cM''[\mathbf{p}_{c,\epsilon,x_\star}]$ plays an important role in the stability analysis of $\mathbf{p}_{c,\epsilon,x_\star}$. In particular L_0 has a double zero eigenvalue. Under perturbations the eigenvalue splits in two—one stays at zero and the other moves along the real axis. We compute this splitting using the following approach. Let

$$P_\epsilon = \frac{1}{2\pi i}\int_C \frac{1}{\lambda I - L_\epsilon} d\lambda$$

where C is small positively oriented loop about the origin in the complex plane and I is the identity map. P_ϵ defines a projection onto the spectral subspace of any eigenvalues contained in C. We denote this subspace by $\mathcal{E}_\epsilon$. Since L_0 is self-adjoint, P_0 coincides with the orthogonal projection P from $L^2 \times L^2$ onto $\mathcal{E}_0 = \ker L_0$ defined in (12). P_ϵ is analytic in ϵ and commutes with L_ϵ. The operator

$$M_\epsilon(\lambda) = P_0(\lambda I - L_\epsilon)P_\epsilon$$

can be viewed as a linear map from $\mathcal{E}_0$ to $\mathcal{E}_0$, which is to say from $\mathbf{R}^2$ to $\mathbf{R}^2$. If $\det(M_\epsilon(\lambda)) = 0$, then λ is an eigenvalue of L_ϵ. And so we will study this map.

Expanding M_ϵ in a Taylor series yields

$$M_\epsilon(\lambda) = P_0(\lambda I - L_0)P_0 + \epsilon P_0(\lambda I - L_0)P'_0 - \epsilon P_0 L_{0,1} P_0 + O(\epsilon^2)$$

where

$$P'_0 = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} P_\epsilon$$

and

$$L_{0,1} = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} L_\epsilon.$$

We remark that $L_{0,1}$ is self-adjoint.

We claim that $P_0(\lambda I - L_0)P'_0 P_0 = 0$. Since $P_\epsilon P_\epsilon = P_\epsilon$, we have

$$P'_0 P_0 + P_0 P'_0 = P'_0$$
$$P_0 P'_0 = P'_0(I - P_0)$$
$$P_0 P'_0 P_0 = 0.$$

Since $P_0 L_0 = L_0 P_0$, we have shown the above claim.

If we define the isomorphism of $\mathcal{E}_0$ with $\mathbf{R}^2$ by $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 \longmapsto (a_1, a_2)^t$ we see that

$$\det M_\epsilon(\lambda) = \det \left\{\langle \mathbf{v}_i, (\lambda I - L_0)\mathbf{v}_j\rangle - \epsilon\langle \mathbf{v}_i, L_{0,1}\mathbf{v}_j\rangle + O(\epsilon^2)\right\}_{ij}$$

Clearly

$$\{\langle \mathbf{v}_i, (\lambda I - L_0)\mathbf{v}_j\rangle\}_{ij} = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}.$$

We need to compute $\{\langle \mathbf{v}_i, L_{0,1}\mathbf{v}_j\rangle\}_{ij}$. We know that $L_\epsilon \mathbf{p}'_{c,\epsilon,x_\star} = 0$ for all ϵ. If we take the derivative of this equation with respect to ϵ and evaluate at $\epsilon = 0$, we find

$$L_{0,1}\mathbf{p}'_{c,0,x_\star} + L_0\mathbf{r}' = 0$$

where $\mathbf{r} = \frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \mathbf{p}_{c,\epsilon,x_\star}$.

Since $\mathbf{v}_1 \in \mathcal{E}_0$ and L_0 is self-adjoint, if we take the inner product of the above expression with $\mathbf{v}_1$ we have $\langle \mathbf{v}_1, L_{0,1}\mathbf{p}'_{c,0,x_\star}\rangle = 0$. Since $\mathbf{p}_{c,0,x_\star} = h_1\mathbf{v}_1 + h_2\mathbf{v}_2$ where $h_1 = \|q'_{c,0}\|$ and $h_2 = \|r'_{c,0}\|$ this implies that $h_1\langle \mathbf{v}_1, L_{0,1}\mathbf{v}_1\rangle = -h_2\langle \mathbf{v}_1, L_{0,1}\mathbf{v}_2\rangle$. Similarly $h_1\langle \mathbf{v}_2, L_{0,1}\mathbf{v}_1\rangle = -h_2\langle \mathbf{v}_2, L_{0,1}\mathbf{v}_2\rangle$. These relations can be rewritten as

$$\begin{aligned} &\left\langle \begin{pmatrix} q'_{c,0} \\ 0 \end{pmatrix}, L_{0,1}\begin{pmatrix} 0 \\ r'_{c,0} \end{pmatrix} \right\rangle \\ &= -\left\langle \begin{pmatrix} q'_{c,0} \\ 0 \end{pmatrix}, L_{0,1}\begin{pmatrix} q'_{c,0} \\ 0 \end{pmatrix} \right\rangle \\ &= -\left\langle \begin{pmatrix} 0 \\ r'_{c,0} \end{pmatrix}, L_{0,1}\begin{pmatrix} 0 \\ r'_{c,0} \end{pmatrix} \right\rangle. \end{aligned} \tag{16}$$

If we set

$$\kappa \equiv \langle h_1\mathbf{v}_1, L_{0,1}h_2\mathbf{v}_2\rangle$$

then

$$\{\langle \mathbf{v}_i, L_{0,1}\mathbf{v}_j\rangle\}_{ij} = \kappa \begin{pmatrix} -1/h_1^2 & 1/h_1h_2 \\ 1/h_1h_2 & -1/h_2^2 \end{pmatrix}.$$

Note that the definition of κ above is equivalent to that in (10). Therefore

$$\det M_\epsilon(\lambda) = \lambda^2 + \epsilon\kappa\left(1/h_1^2 + 1/h_2^2\right)\lambda + O(\epsilon^2).$$

The eigenvalues are $\lambda = 0$ (exactly) and $\lambda = -\epsilon\kappa\left(1/h_1^2 + 1/h_2^2\right) + O(\epsilon^2)$. Since $\left(1/h_1^2 + 1/h_2^2\right)$ is positive we see that the new eigenvalue splits in the direction opposite to the sign of $\epsilon\kappa$.

A routine calculation shows that we can rewrite κ as

$$\kappa = \frac{1}{4}\left\langle \begin{pmatrix} q'_{c,0} \\ -r'_{c,0} \end{pmatrix}, L_{0,1}\begin{pmatrix} q'_{c,0} \\ -r'_{c,0} \end{pmatrix} \right\rangle.$$

This quantity is linked to the change in energy of the solitary wave if the components are separated. Consider

$$\Delta(\epsilon,\sigma) = E_\epsilon\left[\begin{pmatrix} q_{c,\epsilon}(y+\sigma) \\ r_{c,\epsilon}(y-\sigma) \end{pmatrix}\right] - E_\epsilon\left[\begin{pmatrix} q_{c,\epsilon}(y) \\ r_{c,\epsilon}(y) \end{pmatrix}\right]$$

where $\mathbf{p}_{c,\epsilon,x_\star}(y) = (q_{c,\epsilon}(y), r_{c,\epsilon}(y))^t$. Notice that $\Delta(\epsilon, 0) = 0$ and $\Delta(0,\sigma) = 0$ for any ϵ or σ. Thus $\partial_\epsilon^n\Delta(0,0) = \partial_\sigma^n\Delta(0,0) = 0$ for any n. Moreover, $\partial_\sigma\Delta(\epsilon,0) = 0$ for all ϵ. To check this, first notice that

$$M\left[\begin{pmatrix} q_{c,\epsilon}(y+\sigma) \\ r_{c,\epsilon}(y-\sigma) \end{pmatrix}\right]$$

is constant in σ. And so we have

$$\begin{aligned}
\partial_\sigma \Delta(\epsilon,0) &= \frac{\partial}{\partial\sigma}\Big|_{\sigma=0}\left(E_\epsilon\left[\begin{pmatrix} q_{c,\epsilon}(y+\sigma) \\ r_{c,\epsilon}(y-\sigma)\end{pmatrix}\right] - E_\epsilon\left[\begin{pmatrix} q_{c,\epsilon}(y) \\ r_{c,\epsilon}(y)\end{pmatrix}\right]\right) \\
&= \frac{\partial}{\partial\sigma}\Big|_{\sigma=0}\left(E_\epsilon\left[\begin{pmatrix} q_{c,\epsilon}(y+\sigma) \\ r_{c,\epsilon}(y-\sigma)\end{pmatrix}\right] + cM\left[\begin{pmatrix} q_{c,\epsilon}(y+\sigma) \\ r_{c,\epsilon}(y-\sigma)\end{pmatrix}\right] - E_\epsilon\left[\begin{pmatrix} q_{c,\epsilon}(y) \\ r_{c,\epsilon}(y)\end{pmatrix}\right]\right) \\
&= \left\langle E'_\epsilon\left[\begin{pmatrix} q_{c,\epsilon}(y) \\ r_{c,\epsilon}(y)\end{pmatrix}\right] + cM'\left[\begin{pmatrix} q_{c,\epsilon}(y) \\ r_{c,\epsilon}(y)\end{pmatrix}\right], \begin{pmatrix} q'_{c,\epsilon}(y) \\ -r'_{c,\epsilon}(y)\end{pmatrix}\right\rangle \\
&= \frac{\partial}{\partial\epsilon}\Big|_{\epsilon=0}\left\langle E'_\epsilon\left[\mathbf{p}_{c,\epsilon,x_\star}\right] + cM'\left[\mathbf{p}_{c,\epsilon,x_\star}\right], \begin{pmatrix} q'_{c,\epsilon}(y) \\ -r'_{c,\epsilon}(y)\end{pmatrix}\right\rangle \\
&= 0.
\end{aligned}$$

Thus $\partial_\epsilon^n \partial_\sigma \Delta(0,0) = 0$ for all n. With these facts in hand, expanding Δ in Taylor series results in $\Delta(\epsilon,\sigma) = \frac{\epsilon\sigma^2}{2}\partial_\epsilon\partial_\sigma^2\Delta(0,0) + O(\epsilon^4+\sigma^4)$. Finally, a direct computation shows that $\partial_\epsilon\partial_\sigma^2\Delta(0,0) = 4\kappa$. That is

$$\Delta(\epsilon,\sigma) = 2\sigma^2\epsilon\kappa + O(\epsilon^4+\sigma^4).$$

6 The linearization and its spectrum

We also wish to compute the eigenvalues of the full linearization $\partial_y L_\epsilon = \partial_y\left(E''_\epsilon[\mathbf{p}_{c,\epsilon,x_\star}] + cM''[\mathbf{p}_{c,\epsilon,x_\star}]\right)$. (Once again, $\mathbf{p}_{c,\epsilon}$ is a special case.) Recall that $\partial_y L_0$ has a geometrically double but algebraically quadruple zero eigenvalue. Specifically we have

$$\begin{aligned}
\partial_y L_0 \mathbf{e}_1 = 0, \quad \partial_y L_0 \mathbf{e}_2 = \mathbf{e}_1 \\
\partial_y L_0 \mathbf{e}_3 = 0, \quad \partial_y L_0 \mathbf{e}_4 = \mathbf{e}_3
\end{aligned}$$

where

$$\begin{aligned}
\mathbf{e}_1 &= \begin{pmatrix} q'_{c,0}(x+x_\star) \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} -\partial_c q_{c,0}(x+x_\star) \\ 0 \end{pmatrix} \\
\mathbf{e}_3 &= \begin{pmatrix} 0 \\ r'_{c,0}(x-x_\star) \end{pmatrix}, \quad \mathbf{e}_4 = \begin{pmatrix} 0 \\ -\partial_c r_{c,0}(x-x_\star) \end{pmatrix}.
\end{aligned}$$

Our approach will be the same as in the previous section, but there are a number of complications, the primary one being that zero is embedded in the essential spectrum (which is the imaginary axis). As a result the spectral projection

$$\frac{1}{2\pi i}\int_C \frac{1}{\lambda I - \partial_y L_\epsilon} d\lambda$$

is not well defined since any curve C about the origin necessarily crosses the imaginary axis.

We deal with this difficulty by studying $\partial_y L_\epsilon$ in an exponentially weighted space (see [18] for an overview of this approach). Let $L_a^2 = \left\{ v \middle| e^{ay} v \in L^2 \right\}$. The spectrum of $\partial_y L_\epsilon$ in $L_a^2 \times L_a^2$ is equal to the spectrum of the "conjugated" operator $A_{a,\epsilon} = e^{ay}\partial_y L_\epsilon e^{-ay}$ in $L^2 \times L^2$. The advantage here is that while the eigenvalues of $A_{a,0}$ remain at zero, the essential spectrum is shifted off of the axis.

That the eigenvalues do not move is straight-forward. We have

$$\begin{aligned}
A_{a,0}\mathbf{g}_1 = 0, \quad A_{a,0}\mathbf{g}_2 = \mathbf{g}_1 \\
A_{a,0}\mathbf{g}_3 = 0, \quad A_{a,0}\mathbf{g}_4 = \mathbf{g}_3
\end{aligned}$$

where $\mathbf{g}_j = e^{ay}\mathbf{e}_j$, $j = 1...4$. Of course, we must require that these functions are still in $L^2 \times L^2$; since all the functions are exponentially decaying at infinity—this simply puts limits on our choices for a.

We can compute the essential spectrum of $A_{a,\epsilon}$, which we denote $\sigma_{\text{ess}}(A_{a,\epsilon})$. Let $\mathbf{p}_{c,\epsilon,x_\star} = (q_{c,\epsilon}, r_{c,\epsilon})^t$.

Then

$$L_\epsilon = \begin{pmatrix} c - K_1 - F_1''(q_{c,\epsilon}) - \epsilon\partial_u^2 H(\mathbf{p}_{c,\epsilon,x_\star}) & -\epsilon\left(K_3 + \partial_u\partial_v H(\mathbf{p}_{c,\epsilon,x_\star})\right) \\ -\epsilon\left(K_3 + \partial_u\partial_v H(\mathbf{p}_{c,\epsilon,x_\star})\right) & c - K_2 - F_2''(r_{c,\epsilon}) - \epsilon\partial_v^2 H(\mathbf{p}_{c,\epsilon,x_\star}) \end{pmatrix}.$$

Since $\mathbf{p}_{c,\epsilon,x_\star}$ decays exponentially, we find that $A_{a,\epsilon}$ is a compact perturbation of the operator $B_{a,\epsilon} = e^{ay}\partial_y \tilde{L}_\epsilon e^{-ay}$ where

$$\tilde{L}_\epsilon = \begin{pmatrix} c - K_1 & -\epsilon K_3 \\ -\epsilon K_3 & c - K_2 \end{pmatrix}. \tag{17}$$

The essential spectrum of $\partial_y \tilde{L}_\epsilon$ consists of those points λ such that $\partial_y \tilde{L}_\epsilon - \lambda I$ fails to have a bounded inverse. We can explicitly compute the inverse of this operator by means of the Fourier transform; $(\partial_y \tilde{L}_\epsilon - \lambda I)^{-1}\mathbf{u}$ is the inverse Fourier transform of

$$\frac{1}{ik}\begin{pmatrix} c - \widehat{K}_1(k) - \lambda & -\epsilon\widehat{K}_3(k) \\ -\epsilon\widehat{K}_3(k) & c - \widehat{K}_2(k) - \lambda \end{pmatrix}^{-1} \widehat{\mathbf{u}}$$

where $\widehat{K}_1(k) = k^2$, $\widehat{K}_2(k) = c_1 - c_2k^2$, $\widehat{K}_3(k) = c_3 - c_4k^2$. (That is $\widehat{K_j f}(k) = \widehat{K}_j(k)\widehat{f}(k)$.) The inverse operator is unbounded if, for some $k \in \mathbf{R}$, λ is an eigenvalue of the matrix

$$ik\begin{pmatrix} c - \widehat{K}_1(k) & -\epsilon\widehat{K}_3(k) \\ -\epsilon\widehat{K}_3(k) & c - \widehat{K}_2(k) \end{pmatrix}.$$

Carrying out this computation shows that

$$\sigma_{\text{ess}}(\partial_y L_\epsilon) = \left\{ \frac{ik}{2}\left(2c - \widehat{K}_1(k) - \widehat{K}_2(k) \pm \sqrt{\left(\widehat{K}_1(k) - \widehat{K}_2(k)\right)^2 + 4\epsilon^2\widehat{K}_3(k)^2}\right) \text{ where } k \in \mathbf{R} \right\}.$$

Since $\widehat{K}_1$ and $\widehat{K}_3$ are quadratic in k we see that the essential spectrum of $\partial_y L_\epsilon$ is the entire imaginary axis.

Noting that $e^{ay}\partial_y e^{-ay} f = (\partial_y - a)f$, repeating the above argument leads to

$$\sigma_{\text{ess}}(A_{a,\epsilon}) = \left\{ \frac{il}{2}\left(2c - \widehat{K}_1(l) - \widehat{K}_2(l) \pm \sqrt{\left(\widehat{K}_1(l) - \widehat{K}_2(l)\right)^2 + 4\epsilon^2\widehat{K}_3(l)^2}\right) \text{ where } l = k + ia,\ k \in \mathbf{R} \right\}. \tag{18}$$

We made the assumption that c_2 is positive and as a consequence one can check that if a is small and positive, then $\sigma_{\text{ess}}(A_{a,\epsilon})$ is shifted into the stable left half plane. (Apart from the zero eigenvalues, there are no eigenvalues of $A_{a,0}$ on the imaginary axis, another result of [18].)

In this way we have isolated the zero eigenvalues of $A_{a,0}$ from the essential spectrum and therefore we can employ the perturbative techniques used above to compute the splitting of the zero eigenvalue. Let

$$\Pi_\epsilon = \frac{1}{2\pi i}\int_C \frac{1}{\lambda I - A_{a,\epsilon}} d\lambda$$

where C is a small positively oriented loop about the origin. Π_ϵ is the projection of $L^2 \times L^2$ onto the spectral subspace associated to any eigenvalues contained in C. Let $\mathcal{F}_\epsilon$ be this subspace—in particular $\mathcal{F}_0 = \text{span}\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3, \mathbf{g}_4\}$. Π_ϵ is analytic in ϵ and commutes with $A_{a,\epsilon}$. Let

$$N_\epsilon(\lambda) = \Pi_0(\lambda I - A_{a,\epsilon})\Pi_\epsilon.$$

N_ϵ defines a linear map from $\mathcal{F}_0$ to $\mathcal{F}_0$, or equivalently from $\mathbf{R}^4$ to $\mathbf{R}^4$. As above, if $\det N_\epsilon(\lambda) = 0$ then λ is an eigenvalue of $A_{a,\epsilon}$. If we expand $N_\epsilon(\lambda)$ in ϵ we find

$$N_\epsilon(\lambda) = \Pi_0(\lambda I - A_{a,0})\Pi_0 - \epsilon \Pi_0 A'_{a,0} \Pi_0 + O(\epsilon^2)$$

where $A'_{a,0} = \left.\frac{\partial}{\partial \epsilon}\right|_{\epsilon=0} A_{a,\epsilon}$. (As with the expansion of $M_\epsilon(\lambda)$ above, there is an additional term in the expansion which vanishes.)

We need to determine Π_0. $A_{a,0}$ is not self-adjoint and so Π_0 does not correspond to an orthogonal projection. Instead, Π_0 is identical to the projection onto $\mathcal{F}_0$ obtained by taking inner products with eigenfunctions of the adjoint of $A_{a,0}$,

$$A^\dagger_{a,0} = -e^{-ay} L_0 \partial_y e^{ay}.$$

We first discuss the kernel of $A^\dagger_{0,0} = -L_0 \partial_y$. Notice that

$$\begin{aligned} (-L_0\partial_y)\partial_y^{-1}\mathbf{e}_1 = 0, \quad (-L_0\partial_y)\partial_y^{-1}\mathbf{e}_2 = -\partial_y^{-1}\mathbf{e}_1 \\ (-L_0\partial_y)\partial_y^{-1}\mathbf{e}_3 = 0, \quad (-L_0\partial_y)\partial_y^{-1}\mathbf{e}_4 = -\partial_y^{-1}\mathbf{e}_3. \end{aligned}$$

Formally, we see that the functions $\partial_y^{-1}\mathbf{e}_j$, $j = 1...4$ are in the generalized kernel. Since $\mathbf{e}_1 = (q'_{c,0}, 0)^t$, $\partial_y^{-1}\mathbf{e}_1 = (q_{c,0}, 0)^t$ is a perfectly well-defined function in $L^2 \times L^2$. The same holds for $\mathbf{e}_3$. However, it is not generally going to be the case that $\partial_y^{-1}\mathbf{e}_2$ and $\partial_y^{-1}\mathbf{e}_4$ will be square integrable. Nevertheless we define the functions

$$\begin{aligned} \mathbf{e}_1^* = a_1 \begin{pmatrix} \partial_y^{-1}\partial_c q_{c,0} \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} q_{c,0} \\ 0 \end{pmatrix}, \quad \mathbf{e}_2^* = a_1 \begin{pmatrix} q_{c,0} \\ 0 \end{pmatrix} \\ \mathbf{e}_3^* = b_1 \begin{pmatrix} 0 \\ \partial_y^{-1}\partial_c r_{c,0} \end{pmatrix} + b_2 \begin{pmatrix} 0 \\ r_{c,0} \end{pmatrix}, \quad \mathbf{e}_4^* = b_1 \begin{pmatrix} 0 \\ r_{c,0} \end{pmatrix} \end{aligned}$$

where

$$\partial_y^{-1} \cdot = \int_{-\infty}^{y} \cdot \, dx.$$

The constants a_1, a_2, b_1, b_2 are taken so that

$$\langle \mathbf{e}_i^*, \mathbf{e}_j \rangle = \delta_{ij}$$

with δ_{ij} the Kronecker delta. They are

$$\begin{aligned} a_1 &= -\frac{2}{\partial_c \int q_{c,0}^2}, \quad a_2 = \frac{a_1^2 (\int_{-\infty}^{\infty} \partial_c q_{c,0})^2}{2} \\ b_1 &= -\frac{2}{\partial_c \int r_{c,0}^2}, \quad b_2 = \frac{b_1^2 (\int_{-\infty}^{\infty} \partial_c r_{c,0})^2}{2}. \end{aligned}$$

The functions $\mathbf{e}_i^*$ satisfy

$$\begin{aligned} (-L_0\partial_y)\mathbf{e}_2^* = 0, \quad (-L_0\partial_y)\mathbf{e}_1^* = \mathbf{e}_2^* \\ (-L_0\partial_y)\mathbf{e}_4^* = 0, \quad (-L_0\partial_y)\mathbf{e}_3^* = \mathbf{e}_4^* \end{aligned}$$

and as such are (formally) a set which spans the generalized kernel of the $-L_0\partial_y$.

If we set $\mathbf{g}_j^* = e^{-ay}\mathbf{e}_j^*$, $j = 1...4$ then

$$\begin{aligned} A^\dagger_{a,0}\mathbf{g}_2^* = 0, \quad A^\dagger_{a,0}\mathbf{g}_1^* = \mathbf{g}_2^* \\ A^\dagger_{a,0}\mathbf{g}_4^* = 0, \quad A^\dagger_{a,0}\mathbf{g}_3^* = \mathbf{g}_4^* \end{aligned}$$

Moreover the functions $\mathbf{g}_i^*$ are in $L^2 \times L^2$ if $a > 0$. The exponential weight e^{-ay} leads to convergence for y large and positive. On the other hand, we have defined ∂_y^{-1} as a definite integral which begins at $-\infty$. This fact, together with the exponential decay of the functions we are considering leads to the convergence for $y < 0$. Note that

$$\langle \mathbf{g}_i^*, \mathbf{g}_j \rangle = \delta_{ij}.$$

With this choice for the adjoint eigenfunctions the projection Π_0 takes the form

$$\Pi_0 \mathbf{f} = \sum \langle \mathbf{g}_j^*, \mathbf{f} \rangle \mathbf{g}_j.$$

We define the isomorphism from $\mathcal{F}_0$ to $\mathbf{R}^4$ by $a\mathbf{g}_1 + b\mathbf{g}_2 + c\mathbf{g}_3 + d\mathbf{g}_3 \longmapsto (a, b, c, d)^t$. Therefore

$$\det N_\epsilon(\lambda) = \det \left\{ \langle \mathbf{g}_i^*, (\lambda I - A_{a,0}) \mathbf{g}_j \rangle - \epsilon \langle \mathbf{g}_i^*, A'_{a,0} \mathbf{g}_j \rangle + O(\epsilon^2) \right\}_{ij}.$$

It is clear that

$$\left\{ \langle \mathbf{g}_i^*, (\lambda I - A_{a,0})\, \mathbf{g}_j \rangle \right\}_{ij} = \begin{pmatrix} \lambda & -1 & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & \lambda & -1 \\ 0 & 0 & 0 & \lambda \end{pmatrix}$$

and $\Pi_0 A'_0 \Pi_0$ corresponds to the matrix

$$\begin{aligned} \left\{ \langle \mathbf{g}_i^*, A'_{a,0} \mathbf{g}_j \rangle \right\}_{ij} &= \quad \left\{ \langle \mathbf{e}_i^*, \partial_y L_{0,1} \mathbf{e}_j \rangle \right\}_{ij} \\ &= -\left\{ \langle \partial_y \mathbf{e}_i^*, L_{0,1} \mathbf{e}_j \rangle \right\}_{ij}. \end{aligned}$$

Note that this matrix has no dependence on the exponential weight a. A consequence of this is that eigenvalues of $A_{a,\epsilon}$ do not depend on a at leading order in ϵ, a fact we will exploit later.

We now evaluate the inner products, for example

$$\begin{aligned} \langle \partial_y \mathbf{e}_1^*, L_{0,1} \mathbf{e}_1 \rangle &= \left\langle a_1 \begin{pmatrix} \partial_c q_{c,0} \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} q'_{c,0} \\ 0 \end{pmatrix}, L_{0,1} \begin{pmatrix} q'_{c,0} \\ 0 \end{pmatrix} \right\rangle \\ &= a_1 \mu_1(x_\star) - a_2 \kappa \end{aligned}$$

where κ is as in (10) and

$$\mu_1(x_\star) = \left\langle \begin{pmatrix} \partial_c q_{c,0} \\ 0 \end{pmatrix}, L_{0,1} \begin{pmatrix} q'_{c,0} \\ 0 \end{pmatrix} \right\rangle.$$

Recall that we are linearizing about the solutions $\mathbf{p}_{c,\epsilon,x_\star}$ and therefore $L_{0,1}$ has an implicit dependence on $x_\star$. If $x_\star$ is zero, then L_ϵ maps even functions to even functions and odd to odd. So does $L_{0,1}$, thus $\mu_1(0) = 0$ since it is an inner product of an even and an odd function.

Computing the remainder of the inner products shows that

$$\left\{ \langle \partial_y \mathbf{e}_i^*, L_{0,1} \mathbf{e}_j \rangle \right\}_{ij} = \begin{pmatrix} a_1\mu_1 - a_2\kappa & -a_1\nu_1 - a_2\mu_1 & a_1\mu_2 + a_2\kappa & -a_1\nu_2 - a_2\mu_3 \\ -a_1\kappa & -a_1\mu_1 & a_1\kappa & -a_1\mu_3 \\ b_1\mu_3 + b_2\kappa & -b_1\nu_2 - b_2\mu_2 & b_1\mu_4 - b_2\kappa & -b_1\nu_3 - b_2\mu_4 \\ b_1\kappa & -b_1\mu_2 & -b_1\kappa & -b_1\mu_4 \end{pmatrix}$$

where

$$\mu_2(x_\star) = \left\langle \begin{pmatrix} \partial_c q_{c,0} \\ 0 \end{pmatrix}, L_{0,1} \begin{pmatrix} 0 \\ r'_{c,0} \end{pmatrix} \right\rangle$$
$$\mu_3(x_\star) = \left\langle \begin{pmatrix} 0 \\ \partial_c r_{c,0} \end{pmatrix}, L_{0,1} \begin{pmatrix} q'_{c,0} \\ 0 \end{pmatrix} \right\rangle$$
$$\mu_4(x_\star) = \left\langle \begin{pmatrix} 0 \\ \partial_c r_{c,0} \end{pmatrix}, L_{0,1} \begin{pmatrix} 0 \\ r'_{c,0} \end{pmatrix} \right\rangle$$
$$\nu_1(x_\star) = \left\langle \begin{pmatrix} \partial_c q_{c,0} \\ 0 \end{pmatrix}, L_{0,1} \begin{pmatrix} \partial_c q_{c,0} \\ 0 \end{pmatrix} \right\rangle$$
$$\nu_2(x_\star) = \left\langle \begin{pmatrix} \partial_c q_{c,0} \\ 0 \end{pmatrix}, L_{0,1} \begin{pmatrix} 0 \\ \partial_c r_{c,0} \end{pmatrix} \right\rangle$$
$$\nu_3(x_\star) = \left\langle \begin{pmatrix} 0 \\ \partial_c r_{c,0} \end{pmatrix}, L_{0,1} \begin{pmatrix} 0 \\ \partial_c r_{c,0} \end{pmatrix} \right\rangle.$$

Note that $\mu_i(0) = 0$ for $i = 2...4$ for the same reasons $\mu_1(0) = 0$.

At this point, we could explicitly compute $\det N_\epsilon(\lambda)$, but this results in a rather complicated expression. Instead, since $\partial_y L_\epsilon$ always has an algebraically double zero eigenvalue regardless of ϵ, we know that

$$\det N_\epsilon(\lambda) = \lambda^2 \left(\lambda^2 + B\lambda + C\right).$$

As B is the coefficient multiplying the $O(\lambda^3)$ term in the determinant of a four-by-four matrix, it is simply the trace of $N_\epsilon(0)$. Tedious, but routine, computations show that

$$C = -\epsilon\left(a_1\kappa + b_1\kappa\right) + O(\epsilon^2).$$

Therefore

$$\begin{aligned} \det N_\epsilon(\lambda) =& \det\left\{\langle \mathbf{g}_i^*, (\lambda - A_{a,0}I)\mathbf{g}_j\rangle - \epsilon\langle \mathbf{g}_i^*, A'_{a,0}\mathbf{g}_j\rangle\right\}_{ij} \\ =& \lambda^2\left(\lambda^2 - \epsilon\kappa(a_2+b_2)\lambda - \epsilon\kappa(a_1+b_1) + O(\epsilon^2)\right) \end{aligned}$$

which has zeros at $\lambda = 0$ and

$$\lambda = \frac{1}{2}(a_2+b_2)\epsilon\kappa \pm \sqrt{(a_1+b_1)\,\epsilon\kappa + O(\epsilon^2)}.$$

Hypothesis 2 implies that a_1 and b_1 are negative and $a_2 + b_2$ is manifestly positive. Therefore

1. if $\epsilon\kappa < 0$ then the term appearing in the square root is positive and we see a positive real eigenvalue of $O(\sqrt{|\epsilon|})$ is produced. Therefore the solitary wave is linearly unstable.
2. if $\epsilon\kappa > 0$ then the term appearing in the square root is negative. Thus the real part of the eigenvalues is determined by the sign of $\frac{1}{2}(a_2+b_2)\epsilon\kappa$, which is positive. And so the solitary wave is linearly unstable in this case as well!

Notice that in the first case we expect the instability will grow at a rate $O(\sqrt{|\epsilon|})$ while in the second case we expect the instability to grow at the slower rate $O(\epsilon)$ and to also oscillate. This slowly growing oscillatory instability is the origin of the leapfrogging behavior described in previous literature. This concludes the proof of Theorem 15. As we stated in Corollary 16, it happens that the unstable eigenvalues in the right half plane persist if we set $a = 0$. We prove this now.

We have shown that, for a sufficiently small, there is a complex number λ in the right half plane and a function $\mathbf{u}_a \in L^2_a \times L^2_a$ such that $\partial_y L_\epsilon \mathbf{u}_a = \lambda \mathbf{u}_a$. Recall that λ is, to leading order in ϵ, independent of a. We claim that $\mathbf{u}_a$ is in fact in the space $L^2 \times L^2$. Since $a > 0$,

$\int_0^\infty |\mathbf{u}_a(y)|^2 dy < \int_0^\infty |e^{ay}\mathbf{u}_a(y)|^2 dy < \infty$. It is not clear that $\mathbf{u}_a$ decays fast enough as $y \to -\infty$ for $\int_{-\infty}^0 |\mathbf{u}_a(y)|^2 dy$ to converge.

However, $\mathbf{u}_a$ is a solution of a linear differential equation and thus we have more information available on its behavior at spatial infinity. Since $\mathbf{p}_{c,\epsilon,x_\star}$ decays exponentially fast, $\mathbf{u}_a$ decays or grows at rates equivalent to those of solutions of

$$(\partial_y \tilde{L}_\epsilon - \lambda I)\mathbf{v} = 0$$

where $\tilde{L}_\epsilon$ as in (17). A direct computation using the definitions of the operators K_i shows that solutions of this ODE behave at spatial infinity like $e^{\mu y}$ where μ is one of the roots of

$$\left(\mu^3 - c\mu + \lambda\right)\left(c_2\mu^3 + (c_1 - c)\mu + \lambda\right) - \epsilon^2(c_4\mu^3 + c_3\mu)^2 = 0. \tag{19}$$

We claim that if λ has non-zero real part (as is the case), then this equation has no purely imaginary solutions. We check this as follows. Suppose that $\mu = i\beta$ with $\beta \in \mathbf{R}$ is a solution. Then we can rewrite (19) as

$$(i\beta_1 + \lambda)(i\beta_2 + \lambda) + \epsilon^2\beta_3^2 = 0$$

for appropriate real numbers β_1, β_2 and β_3. The equation is quadratic in λ, so we have

$$\lambda = \frac{1}{2}\left(-i(\beta_1 + \beta_2) \pm \sqrt{-(\beta_1 - \beta_2)^2 - 4\epsilon^2\beta_3^2}\right).$$

Thus μ purely imaginary implies λ is as well.

Since the numbers μ must have non-zero real parts, $\mathbf{u}_a$ either grows or decays at an exponential rate as $y \to -\infty$. However, $\mathbf{u}_a \in L^2_a \times L^2_a$ precludes the possibility that it grows without bound. Suppose that for y large and negative $|\mathbf{u}_a(y)| = O(|e^{\mu y}|)$ where μ has negative real part. Thus $|e^{ay}\mathbf{u}_a(y)| = O(|e^{(\mu+a)y}|)$ as $y \to -\infty$. Since $\mathbf{u}_a \in L^2_a \times L^2_a$, this implies $\mu + a$ has negative real part. But a is arbitrarily close to zero and μ is fixed. Thus $\mathbf{u}_a$ cannot grow for $y << 0$, which in turn implies it decays exponentially there. Therefore $\mathbf{u}_a \in L^2 \times L^2$ and λ is in the spectrum of $\partial_y L_\epsilon$ on $L^2 \times L^2$.

7 Illustrative examples and numerical simulations.

In this section we carry out some calculations and perform several numerical studies to demonstrate our principal results. We will be working with the system:

$$\begin{aligned} u_t + \left(u_{xx} + u^2 + \epsilon\left(\frac{1}{2}v^2 + uv\right)\right)_x &= 0 \\ v_t + \left(v_{xx} + v^2 + \epsilon\left(\frac{1}{2}u^2 + uv\right)\right)_x &= 0. \end{aligned} \tag{20}$$

This system is of Gear-Grimshaw type and has the added feature that the equation is invariant under the exchange of u and v. This latter fact allows us to compute that $\mathbf{p}_{c,\epsilon}(y) = q_{c,\epsilon}(y)\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ where

$$q_{c,\epsilon}(y) = \frac{3c}{2 + 3\epsilon}\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}y\right)$$

provided $\epsilon > -2/3$ and $c > 0$. Similarly, we find that

$$L_{0,1} = q_{c,0}\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$$

and

$$\kappa = \left\langle \left(\begin{array}{c} q'_{c,0} \\ 0 \end{array} \right), L_{0,1} \left(\begin{array}{c} 0 \\ q'_{c,0} \end{array} \right) \right\rangle$$
$$= -2 \int \left(q'_{c,0}(x) \right)^2 q_{c,0}(x) dx.$$

Clearly $\kappa < 0$ and we expect that for $\epsilon < 0$ we should observe the leapfrogging instability.

We numerically simulate solutions to (20) using an adaptation of a pseudo-spectral scheme developed by Li & Sattinger in [19]. In our simulations, we let $c = 1$ and take as initial conditions $u(x, 0) = 0.95q_{1,\epsilon}(x)$, $v(x, 0) = q_{1,\epsilon}(x)$, a slight perturbation of $\mathbf{p}_{1,\epsilon}(x)$. We repeat the simulations for sundry values of ϵ. To better demonstrate the leapfrogging phenomenon, we study the motion of the centers of mass of u and v, which we denote $\phi_u(t)$ and $\phi_v(t)$. Specifically $\phi_u(t) = \frac{1}{\|u(t)\|^2_{L^2}} \int xu^2(x, t)dt$ and $\phi_v(t) = \frac{1}{\|v(t)\|^2_{L^2}} \int xv^2(x, t)dt$. If the initial condition were precisely $\mathbf{p}_{1,\epsilon}$, these would both be identically t.

In Figure 3 we plot, for $\epsilon = -0.2$, u and v vs. x at a sequence of times covering roughly one oscillation of the leapfrogging. Note the radiative dispersion to the left of the wave which, though small, forms at about $t = 3$. In Figure 4 we plot $\phi_u(t) - t$ and $\phi_v(t) - t$ versus t for $\epsilon = -0.2$ and $\epsilon = -0.1$. Several features stand out. The first is the growing oscillation between the phases, *i.e.* the leapfrogging behavior predicted by the linear theory. Second, the frequency of this oscillation increases with ϵ. Finally, notice that there is an overall decrease in the phase speed of each component. Heuristically, this is due to the radiation; the dispersion decreases the amount of momentum available to the wave, and thus slows it down.

If one runs the simulation for long times, the amplitude of the oscillation grows while simultaneously more radiation is created. Eventually the oscillations grow so large that the two components cease their interaction. That is to say, the leapfrogging behavior is transient. In fact, the solution appears to become the superposition of waves of $\mathbf{t}_{c,\epsilon}$ and $\mathbf{b}_{c,\epsilon}$ type. The production of the dispersive tail ceases as soon as the components separate and the previously produced radiation falls farther and farther behind the two waves. We plot a snapshot of u and v vs. x at $t = 400$ in Figure 6. We plot $\phi_u(t)$ and $\phi_v(t)$ vs. t in Figure 5. (In both instances, we have taken $\epsilon = -0.1$.) We remark that in many dispersive systems (specifically in the KdV equation), radiative effects are a stabilizing influence on solitary waves. In this instance however, it appears that the radiation is a fundamental part of the instability.

Finally, we repeated the above experiments with $\epsilon = 0.1$. In this case, we do not expect any leapfrogging, though the wave remains unstable. Figure 7 plots $\phi_u(t)$ and $\phi_v(t)$ vs. t. Once again the wave separates into the superposition of waves of type $\mathbf{t}_{c,\epsilon}$ and $\mathbf{b}_{c,\epsilon}$, though there is no oscillation—see Figure 9. The waves merely "slide" apart from one another—see Figure 8. Figure 9 is a snapshot of this solution at $t = 150$, well after the waves have separated.

8 Appendix: Alternate proof for existence of asymmetric solitary waves.

In this section we prove the existence of asymmetric solitary waves using an *ad hoc* method which is commonly used in bifurcation and numerical analysis of Hamiltonian systems. The idea is to augment the Hamiltonian equation with a dissipation term, for instance $\gamma \nabla H$, so that for $\gamma \neq 0$, the system is gradient-like and does not possess any small non-equilibrium solutions. The parameter γ then allows one to solve the reduced equation with a standard implicit function theorem; see the proof of the Liapunov Center theorem in [8] for a simple application of this trick. .

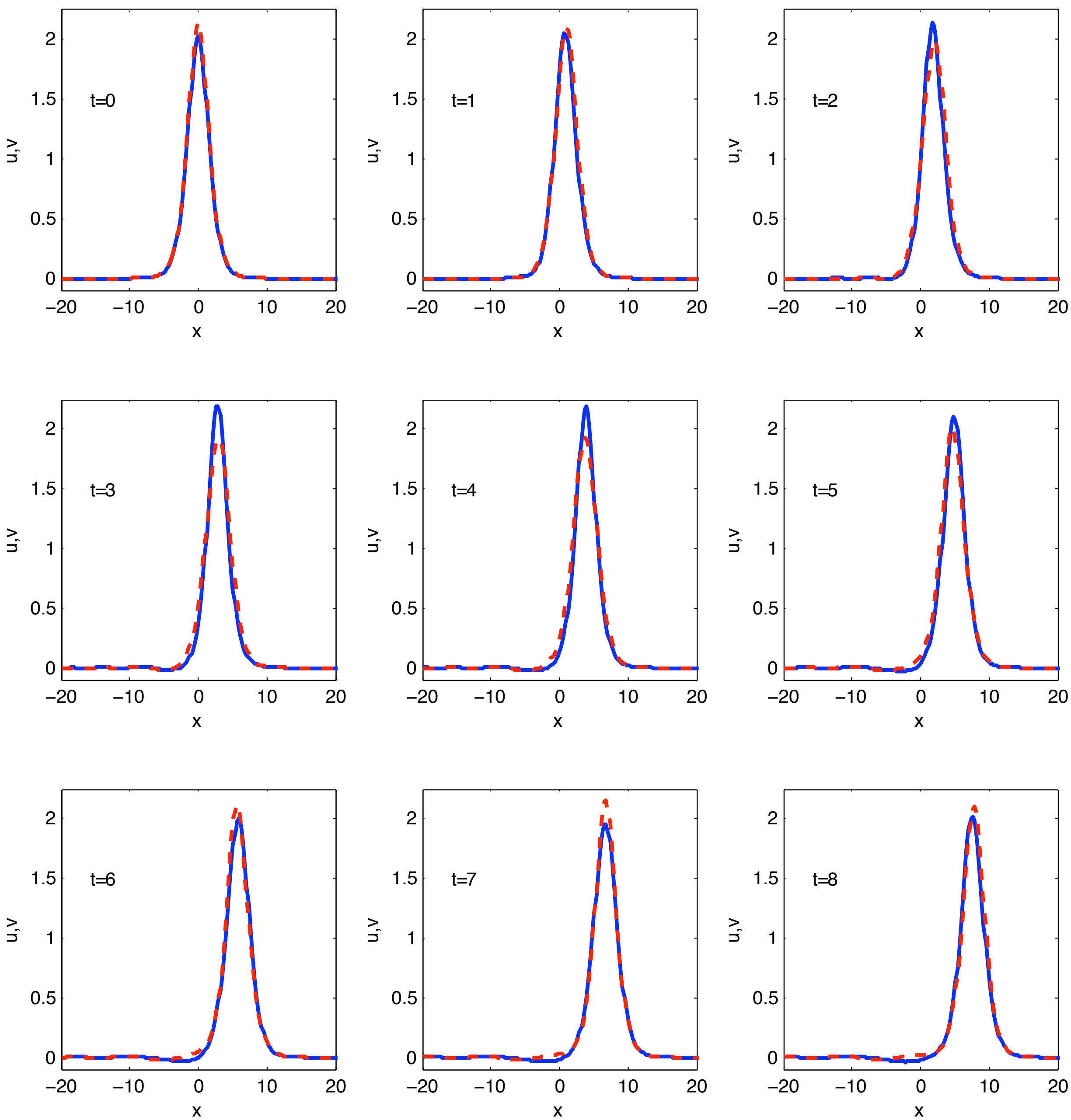

Figure 3: u and v vs. x. Dashed lines are v, solid lines are u. Here, $\epsilon = -0.2$.

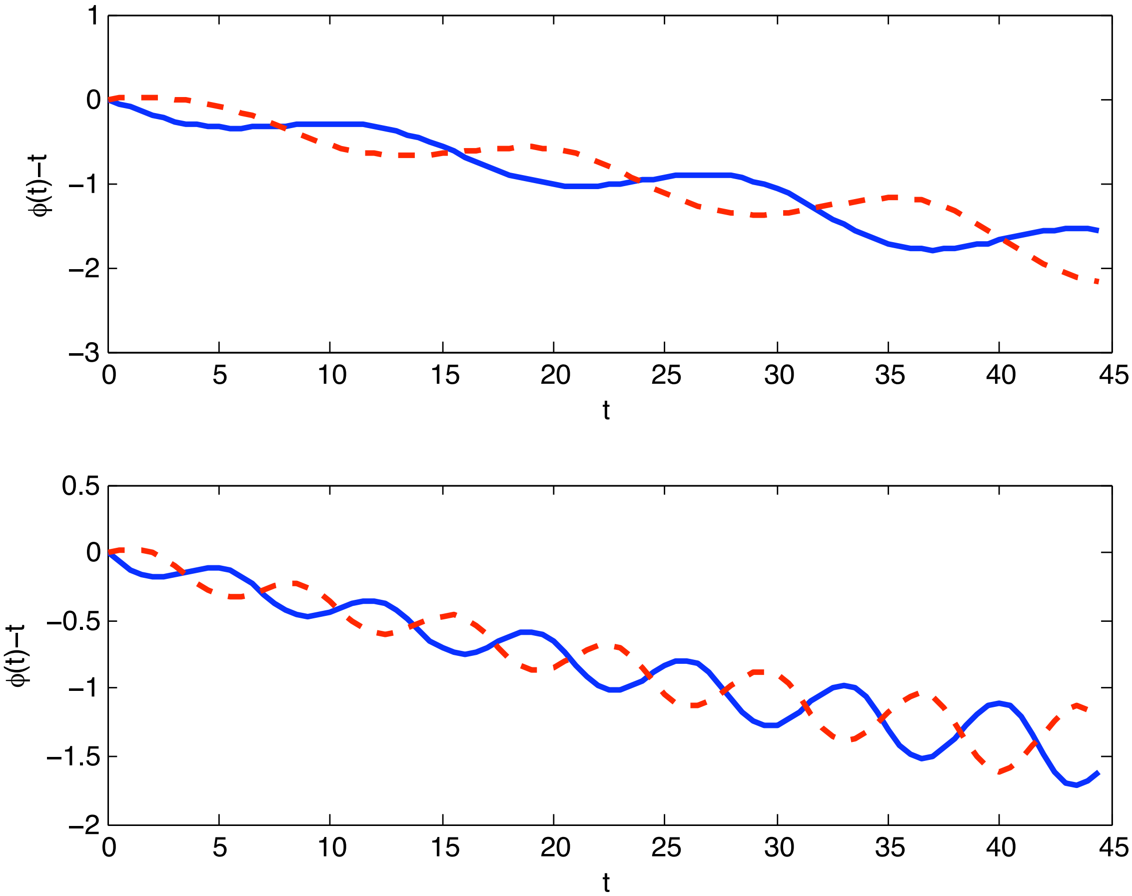

Figure 4: $\phi(t) - t$ vs. t. In the top figure $\epsilon = -0.1$, the bottom $\epsilon = -0.2$.

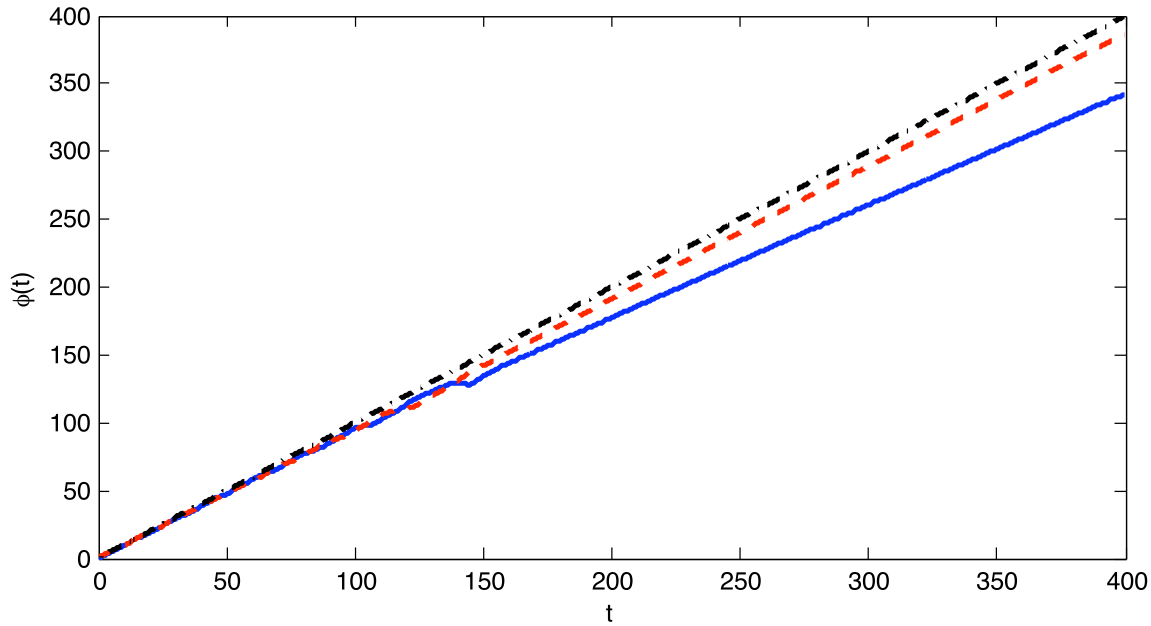

Figure 5: ϕ_u and ϕ_v vs. t. ϕ_u is solid, ϕ_v is dashed. The dashed/dotted line is simply t. Here, $\epsilon = -0.1$.

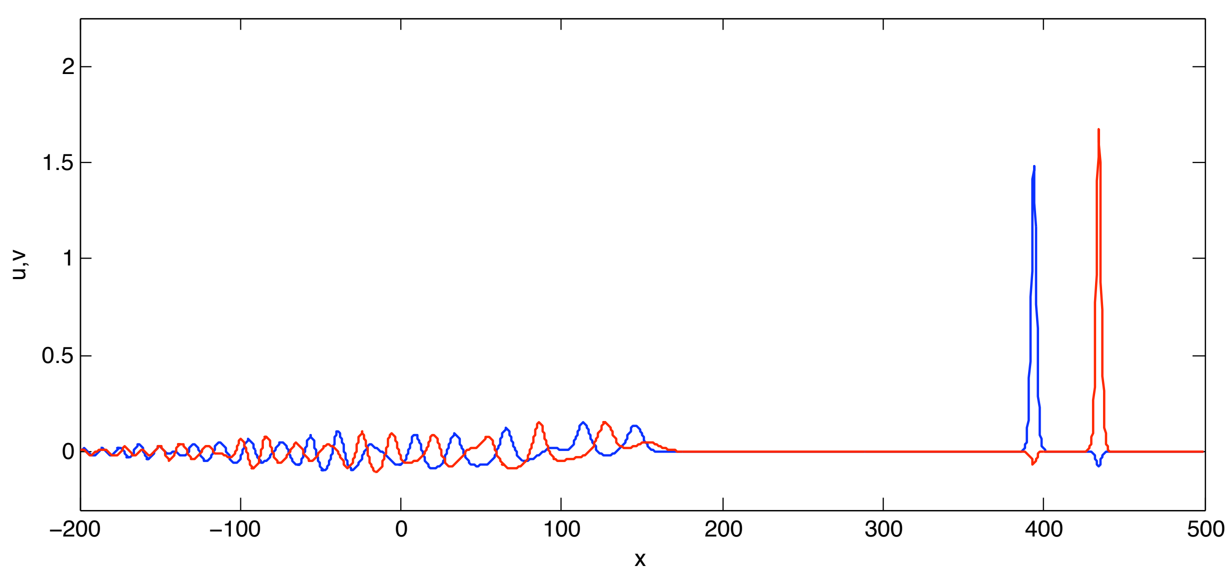

Figure 6: u and v vs. x at $t = 400$ when $\epsilon = -0.1$.

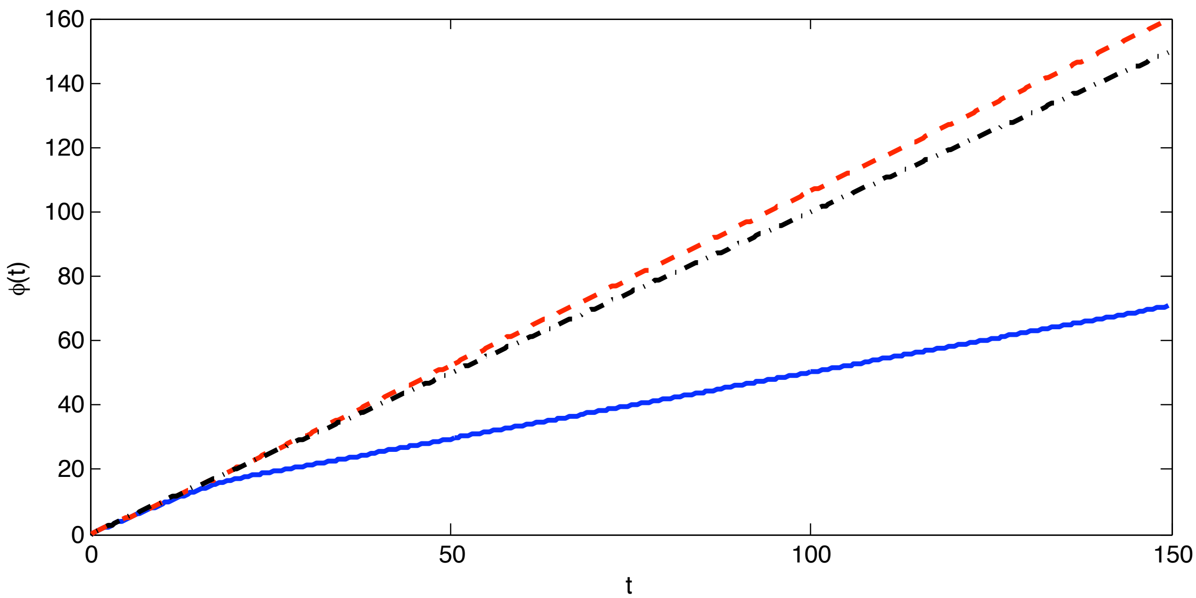

Figure 7: $\phi_u(t)$ and $\phi_v(t)$ vs. t ϕ_u is solid, ϕ_v is dashed. The dashed/dotted line is simply t. Here, $\epsilon = 0.1$.

In our problem, we introduce the dissipation term as a linear damping term. To start with, we employ a Liapunov-Schmidt reduction as in the proof in Section 4. We are looking for $\mathbf{d}(y) \in H^2 \times H^2$ such that $E'_\epsilon[\mathbf{p}_{c,0,x_1} + \mathbf{d}] + c\nabla M[\mathbf{p}_{c,0,x_1} + \mathbf{d}] = 0$. Define the mapping

$$\Phi(\mathbf{d}; \epsilon, x_1, \gamma) = E'_\epsilon[\mathbf{p}_{c,0,x_1} + \mathbf{d}] + cM'[\mathbf{p}_{c,0,x_1} + \mathbf{d}] + \gamma \left(\mathbf{p}'_{c,0,x_1} + \mathbf{d}'\right)$$

Notice that $\Phi(0; 0, x_1, 0) = 0$ for all x_1.

We claim that unless γ is zero, there are no non-trivial solutions to $\Phi = 0$. To see why, suppose that $\mathbf{q} = \mathbf{p}_{c,\epsilon,x_1} + \mathbf{d}$ is such a solution and take the inner product of $\Phi = 0$ with $\mathbf{q}'$. We discover

$$\begin{aligned} 0 =& \langle E'_\epsilon[\mathbf{q}] + cM'[\mathbf{q}] + \gamma \mathbf{q}', \mathbf{q}' \rangle \\ =& \gamma \|\mathbf{q}'\|^2. \end{aligned}$$

Thus either γ is zero or $\mathbf{q}$ is trivial. We remark that adding this term is equivalent to adding a diffusive term to the original system (2). For the time being we allow γ to vary as the presence of an extra parameter will be useful in what follows.

The operator L_0 in (11) is the derivative of Φ with respect to $\mathbf{d}$ and evaluated at $\mathbf{d} = 0$, $\epsilon = 0$, x_1 and $\gamma = 0$. Let $\mathcal{R}$ be the range of this operator, $\mathcal{E}_0$ its kernel of and P the orthogonal projection onto $\mathcal{E}_0$. (All of these coincide with their namesakes in Section 4.) $\Phi(\mathbf{d}; \epsilon, x_1, \gamma) = 0$ if and only if

$$\begin{aligned} P\ \Phi(\mathbf{d}; \epsilon, x_1, \gamma) = 0, &\text{ and} \\ (1 - P)\ \Phi(\mathbf{d}; \epsilon, x_1, \gamma) = 0 & \end{aligned}$$

simultaneously. $1 - P$ is the projection onto $\mathcal{R}$.

$$\Phi(\mathbf{d}; \epsilon, x_1, \gamma) = E'_\epsilon[\mathbf{p}_{c,0,x_1} + \mathbf{d}] + cM'[\mathbf{p}_{c,0,x_1} + \mathbf{d}] + \gamma \left(\mathbf{p}'_{c,0,x_1} + \mathbf{d}'\right)$$

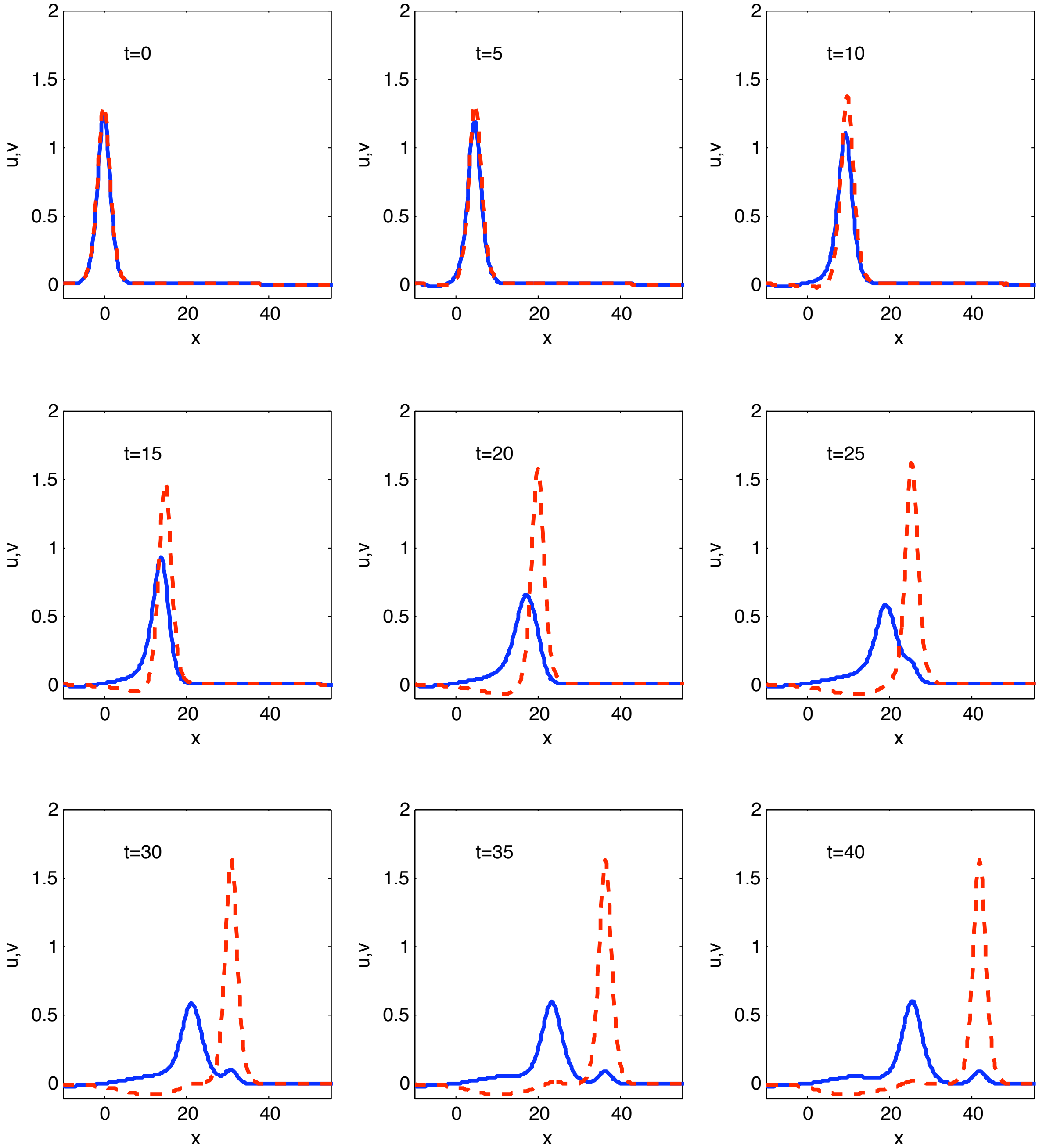

Figure 8: u and v vs. x. Dashed lines are v, solid lines are u. Here, $\epsilon = 0.1$.

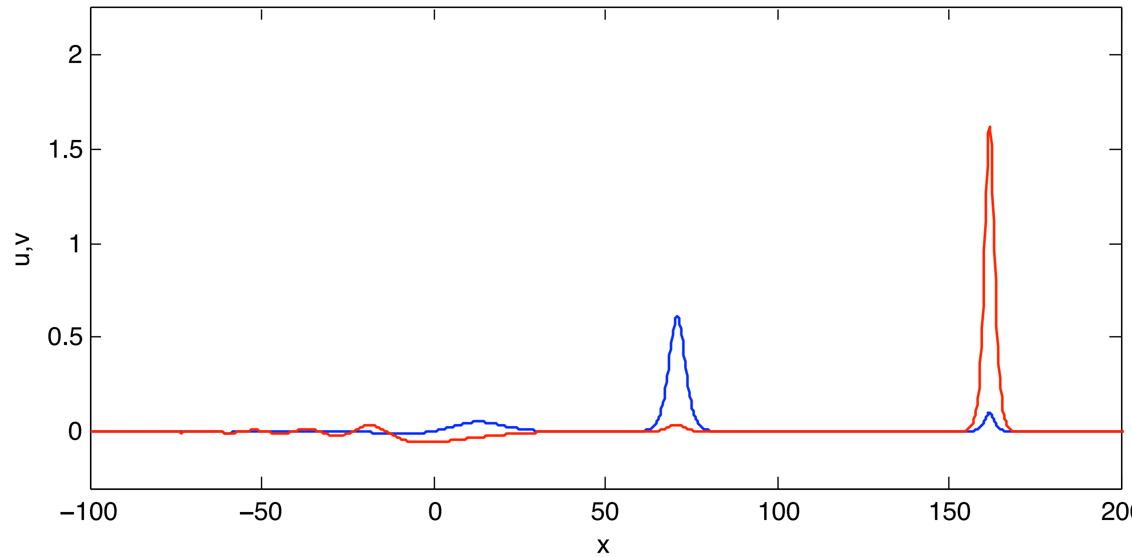

Figure 9: u and v vs. x at $t = 150$ when $\epsilon = 0.1$.

Notice that $\Phi(0; 0, x_1, 0) = 0$ for all x_1.

Also $\mathbf{d} = P\ \mathbf{d} + (1 - P)\ \mathbf{d}$ and so $(1 - P)\ \Phi = 0$ can be rewritten as

$$\Phi_{\mathcal{R}}(\mathbf{d}_{\mathcal{E}_0}, \mathbf{d}_{\mathcal{R}}; \epsilon, x_1, \gamma) \equiv (1 - P)\ \Phi(\mathbf{d}_{\mathcal{E}_0} + \mathbf{d}_{\mathcal{R}}; \epsilon, x_1, \gamma) = 0$$

where $\mathbf{d}_{\mathcal{E}_0} = P\ \mathbf{d}$ and $\mathbf{d}_{\mathcal{R}} = (1-P)\ \mathbf{d}$. $\Phi_{\mathcal{R}}$ is a map from $\mathcal{E}_0 \times \mathcal{R} \times \mathbf{R}^3$ into $\mathcal{R}$. By construction, the derivative of $\Phi_{\mathcal{R}}$ with respect to $\mathbf{d}_{\mathcal{R}}$ has a trivial kernel and is onto. So, by the implicit function theorem we conclude that there is a C^1 map

$$\mathbf{h}_1(\mathbf{d}_{\mathcal{E}_0}, \epsilon, x_1, \gamma)$$

into $\mathcal{R}$ such that

$$\Phi_{\mathcal{E}_0}\ (\mathbf{d}_{\mathcal{E}_0}, \mathbf{h}_1(\mathbf{d}_{\mathcal{E}_0}, \epsilon, x_1\gamma); \epsilon, x_1, \gamma) = 0.$$

(Note that $\mathbf{h}_1(0, 0, x_1, 0) = 0$.)

Solutions to $\Phi = 0$ are in one to one correspondence with solutions of

$$P\ \Phi(\mathbf{d}_{\mathcal{E}_0} + \mathbf{h}_1(\mathbf{d}_{\mathcal{E}_0}, \epsilon, x_1, \gamma); \epsilon, x_1, \gamma) = 0.$$

Recalling that the elements in $\mathcal{E}_0$ are generated by the translation and separation invariances, we will look for solutions where there is no contribution from $\mathcal{E}_0$. To wit, set $\mathbf{h}(\epsilon, x_1, \gamma) = \mathbf{h}_1(0, \epsilon, x_1, \gamma)$. We now must solve

$$\Phi_{\mathcal{E}_0}(\epsilon, x_1, \gamma) \equiv P\ \Phi(\mathbf{h}(\epsilon, x_1, \gamma); \epsilon, x_1, \gamma) = 0.$$

If we expand $\Phi_{\mathcal{E}_0}$ about $\epsilon = 0$, $\gamma = 0$ as a Taylor series we find

$$\Phi_{\mathcal{E}_0}(\epsilon, x_1, \gamma) = P\ \ (L_0\mathbf{h} + \epsilon\partial_\epsilon \Phi(0; 0, x_1, 0) + \gamma\partial_\gamma \Phi(0; 0, x_1, 0)) + O(\epsilon^2 + \gamma^2).$$

Since P is projection onto $\mathcal{E}_0$ and $\mathcal{R} = {\mathcal{E}_0}^{\perp}$, $P\ L_0 = 0$. So we now have

$$P\ \ (\epsilon\partial_\epsilon \Phi(0; 0, x_1, 0) + \gamma\partial_\gamma \Phi(0; 0, x_1, 0)) + O(\epsilon^2 + \gamma^2) = 0. \tag{21}$$

A straight-forward calculation shows that

$$P\ \partial_\gamma \Phi(0;0,x_1,0) = \begin{pmatrix} q'_{c,0}(y+x_1) \\ r'_{c,0}(y-x_1) \end{pmatrix}.$$

Similarly, since

$$\partial_\epsilon \Phi(0;0,x_1,0) = \begin{pmatrix} K_3 r_{c,0}(y-x_1) + \partial_u H(q_{c,0}(y+x_1), r_{c,0}(y-x_1)) \\ K_3 q_{c,0}(y+x_1) + \partial_v H(q_{c,0}(y+x_1), r_{c,0}(y-x_1)) \end{pmatrix}$$

we have

$$P\ \partial_\epsilon \Phi(0;0,x_1,0) = \alpha(x_1) \begin{pmatrix} q'_{c,0}(y+x_1) \\ 0 \end{pmatrix} + \beta(x_1) \begin{pmatrix} 0 \\ r'_{c,0}(y-x_1) \end{pmatrix}$$

where

$$\begin{aligned}
\alpha(x_1) =& \frac{1}{\|q'_{c,0}\|^2} \int \left(K_3 r_{c,0}\left(y-x_1\right) + \partial_u H\left(q_{c,0}\left(y+x_1\right), r_{c,0}\left(y-x_1\right)\right)\right) q'(y+x_1)\ dy \\
\beta(x_1) =& \frac{1}{\|r'_{c,0}\|^2} \int \left(K_3 q_{c,0}\left(y+x_1\right) + \partial_v H\left(q_{c,0}\left(y+x_1\right), r_{c,0}\left(y-x_1\right)\right)\right) r'(y-x_1)\ dy.
\end{aligned}$$

With these, we see that (21) is equivalent to

$$\begin{aligned}
\epsilon\alpha(x_1) + \gamma + O(\epsilon^2+\gamma^2) = 0 \\
\epsilon\beta(x_1) + \gamma + O(\epsilon^2+\gamma^2) = 0,
\end{aligned}$$

or rather

$$\begin{aligned}
\epsilon\left(\alpha(x_1) - \beta(x_1)\right) + O(\epsilon^2+\gamma^2) = 0 \\
\xi(\epsilon, x_1, \gamma) \equiv \epsilon\left(\alpha(x_1) + \beta(x_1)\right) + 2\gamma + O(\epsilon^2+\gamma^2) = 0.
\end{aligned} \tag{22}$$

Notice $\xi(0, x_1, 0) = 0$ and $\partial_\gamma \xi$ evaluated at $\epsilon = \gamma = 0$ is 2. As a consequence there is a C^1 function $g(\epsilon, x_1)$ so that $\xi(\epsilon, x_1, g(\epsilon, x_1)) = 0$ and $g(0, x_1) = 0$. If we did not put in the term γ above, then it would not be clear that we could make this step and solve $\xi = 0$.

We are left only with solving the first equation in (22), which we rewrite as

$$\eta(\epsilon, x_1) \equiv (\alpha(x_1) - \beta(x_1)) + O(\epsilon) = 0.$$

Here we have made use of the fact that $g(\epsilon, x_1)$ is $O(\epsilon)$ and canceled away a power of ϵ. It is clear that we cannot solve this equation unless $\alpha(x_1) = \beta(x_1)$, as there is no hope in that case for the leading order part to cancel with the $O(\epsilon)$ terms. Suppose this condition is met at the point $x_\star$, then $\eta(0, x_\star) = 0$. If $\partial_{x_1}\eta(0, x_\star) = \alpha'(x_\star) - \beta'(x_\star) \neq 0$, then we can once again appeal to the implicit function theorem to assert the existence of a function $\delta(\epsilon)$ so that $\eta(\epsilon, \delta(\epsilon)) = 0$ and $\delta(0) = x_\star$.

Therefore, if $\alpha(x_\star) = \beta(x_\star)$ and $\alpha'(x_\star) \neq \beta'(x_\star)$ then we can construct functions $\delta(\epsilon)$, $g(\epsilon, x_1)$, $\mathbf{h}(\epsilon, x_1, \gamma)$ so that

$$\Phi(\mathbf{h}(\epsilon, \delta(\epsilon), g(\epsilon, \delta(\epsilon))); \epsilon, \delta(\epsilon), g(\epsilon, \delta(\epsilon))) = 0.$$

As we noted earlier, unless $\gamma = 0$ there are no nontrivial solutions of the equation $\Phi = 0$. As a consequence we know that the function g is identically zero. Thus, we have

$$\Phi(\mathbf{h}(\epsilon, \delta(\epsilon), 0); \epsilon, \delta(\epsilon), 0) = 0$$

which is to say that $\mathbf{p}_{c,0,x_\star} + \mathbf{h}(\epsilon, \delta(\epsilon), 0)$ is the profile of a solitary wave solution of (2).

$$\Phi(\mathbf{d};\epsilon,x_1,\gamma) = E'_\epsilon[\mathbf{p}_{c,0,x_1}+\mathbf{d}] + cM'[\mathbf{p}_{c,0,x_1}+\mathbf{d}] + \gamma\left(\mathbf{p}'_{c,0,x_1}+\mathbf{d}'\right)$$

Notice that $\Phi(0;0,x_1,0)=0$ for all x_1.

Finally, it happens that $\|q'_{c,0}\|^2\alpha(x_1) = \|r'_{c,0}\|^2\beta(x_1)$. This fact follows from (once again) the translation invariance of the problem combined with its hamiltonian structure and we leave the details to the reader. Thus the conditions for existence using this method are identically those found in Section 4.

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